E.5 Product-form Queueing Networks

1 Global Balance - Local Balance

- The equations:

$$\forall i \in S: \quad \sum_{j \in S} \pi_j q_{ji} = \pi_i \sum_{j \in S} q_{ij}$$

or:

$$\pi \cdot \mathbf{Q} = 0$$

with the Normalizing condition:

$$\sum_{i \in S} \pi_i = 1$$

are called the **global balance equations**.

- **Global Balance**:

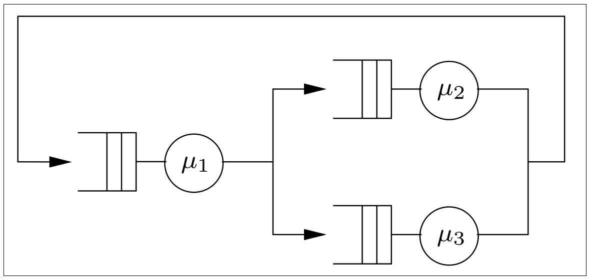
The transition rate **out of** a state of a queueing network
= transition rate **into** this state of the queueing network

- For so called **product-form queueing networks** exists also a **local balance**:

The transition rate **out of** a state of a queueing network due to a departure from **node *i***

= transition rate **into** this state of this queueing network due to an arrival to **node *i***

- Example: Closed queueing network with 3 nodes:



➤ Number of jobs $K = 2$

➤ Service times exp. distributed with: $\mu_1 = 4/\text{sec}$, $\mu_2 = 1/\text{sec}$ und $\mu_3 = 2/\text{sec}$

➤ Strategy: FCFS

➤ Transition probabilities:

$$p_{12} = 0.4, \qquad p_{13} = 0.6$$
$$p_{21} = p_{31} = 1$$

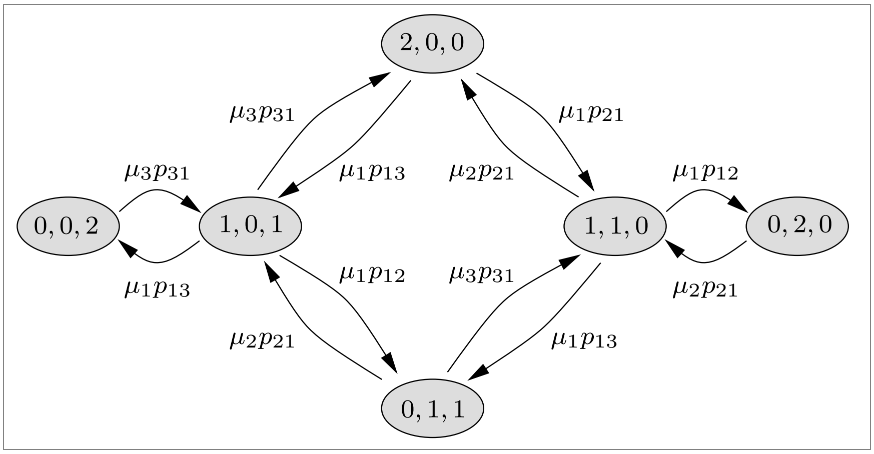
➤ State space of the MC:

$$\{(2, 0, 0),\ (0, 2, 0),\ (0, 0, 2),\ (1, 1, 0),\ (1, 0, 1),\ (0, 1, 1)\}$$

➤ State: $(k_1, k_2, k_3) \rightarrow k_1$ jobs in node 1, k_2 jobs in node 2 und k_3 jobs in node 3

➤ Steady state probability: $\pi(k_1, k_2, k_3)$

◆ State transition diagram:



◆ Global balance equations

$$
\begin{aligned}
&(1) & \pi(2,0,0)(\mu_1 p_{12} + \mu_1 p_{13}) &= \pi(1,0,1)\mu_3 p_{31} + \pi(1,1,0)\mu_2 p_{21}\,, \\
&(2) & \pi(0,2,0)\mu_2 p_{21} &= \pi(1,1,0)\mu_1 p_{12}\,, \\
&(3) & \pi(0,0,2)\mu_3 p_{31} &= \pi(1,0,1)\mu_1 p_{13}\,, \\
&(4) & \pi(1,1,0)(\mu_2 p_{21} + \mu_1 p_{13} + \mu_1 p_{12}) &= \pi(0,2,0)\mu_2 p_{21} + \pi(2,0,0)\mu_1 p_{12} \\
& & &\quad + \pi(0,1,1)\mu_3 p_{31}\,, \\
&(5) & \pi(1,0,1)(\mu_3 p_{31} + \mu_1 p_{12} + \mu_1 p_{13}) &= \pi(0,0,2)\mu_3 p_{31} + \pi(0,1,1)\mu_2 p_{21} \\
& & &\quad + \pi(2,0,0)\mu_1 p_{13}\,, \\
&(6) & \pi(0,1,1)(\mu_3 p_{31} + \mu_2 p_{21}) &= \pi(1,1,0)\mu_1 p_{13} + \pi(1,0,1)\mu_1 p_{12}\,.
\end{aligned}
$$

◆ Local balance equations:

➤ State: $(1, 1, 0)$

– node 2

$$(4') \quad \pi(1,1,0)\cdot \mu_2 \cdot p_{21} = \pi(2,0,0)\cdot \mu_1 \cdot p_{12}$$

– node 1

$$(4'') \quad \pi(1,1,0)\cdot \mu_1 \cdot (p_{13} + p_{12}) = \pi(0,1,1)\cdot \mu_3 \cdot p_{31} + \pi(0,2,0)\cdot \mu_2 \cdot p_{21}$$

By adding these local balance equations, (4') und (4''), we get the global balance equation (4) for state $(1, 1, 0)$.

The equations (1), (2) and (3) are already local balance equations.

The equations (5) and (6) can be splitted to local balance equations corresponding to equation (4):

$$\begin{aligned}
&(5') \quad & \pi(1,0,1)\mu_1(p_{12}+p_{13}) &= \pi(0,1,1)\mu_2 p_{21} + \pi(0,0,2)\mu_3 p_{31}\,, \\
&(5'') \quad & \pi(1,0,1)\mu_3 p_{31} &= \pi(2,0,0)\mu_1 p_{13}\,, \\
&(6') \quad & \pi(0,1,1)\mu_2 p_{21} &= \pi(1,0,1)\mu_1 p_{12}\,, \\
&(6'') \quad & \pi(0,1,1)\mu_3 p_{31} &= \pi(1,1,0)\mu_1 p_{13}\,,
\end{aligned}$$

$$(5') + (5'') = (5) \text{ und } (6') + (6'') = (6)$$

➤ Steady state probabilities:

$$\pi(1,0,1) = \pi(2,0,0)\frac{\mu_1}{\mu_3}p_{13}\,, \qquad \pi(1,1,0) = \pi(2,0,0)\frac{\mu_1}{\mu_2}p_{12}\,,$$
$$\pi(0,0,2) = \pi(2,0,0)\left(\frac{\mu_1}{\mu_3}p_{13}\right)^2\,, \quad \pi(0,2,0) = \pi(2,0,0)\left(\frac{\mu_1}{\mu_2}p_{12}\right)^2\,,$$
$$\pi(0,1,1) = \pi(2,0,0)\frac{\mu_1^2}{\mu_2\mu_3}p_{12}p_{13}\,.$$

➤ Normalizing condition:

$$\pi(2,0,0) = \left[1 + \mu_1\left(\frac{p_{13}}{\mu_3} + \frac{p_{12}}{\mu_2} + \frac{\mu_1 p_{13}^2}{\mu_3^2} + \frac{\mu_1 p_{12}^2}{\mu_2^2} + \frac{\mu_1 p_{12}p_{13}}{\mu_2\mu_3}\right)\right]^{-1}$$

➤ Steady state probabilities:

$$\pi(2,0,0) = \underline{0.103}\,, \quad \pi(0,0,2) = \underline{0.148}\,, \quad \pi(1,0,1) = \underline{0.123}\,,$$
$$\pi(0,2,0) = \underline{0.263}\,, \quad \pi(1,1,0) = \underline{0.165}\,, \quad \pi(0,1,1) = \underline{0.198}\,.$$

➤ From the steady state probabilities the marginal probabilities and from these all performance measures can be calculated.

➤ The local balance equations can be solved much easier than the global balance equations.

➤ The solution is still very complex for greater networks.

Product-form:

- For networks with the "local balance property" a so called **product-form solution** exists:

$$\pi(k_1, k_2, \ldots, k_N) = \frac{1}{G}\left[\pi(k_1)\cdot\pi(k_2)\cdot\ldots\cdot\pi(k_N)\right]$$

  - The steady state probability for the state $(k_1, k_2, \ldots, k_N)$ is the product of the marginal probabilities $\pi(k_1)$, $\pi(k_2)$, ... , $\pi(k_N)$ for the single nodes.
  - The normalizing constant G can be obtained from the normalizing condition (Sum of all steady state probabilities is equal to 1).
  - Queueing networks with the local balance property are called **product-form queueing networks**.

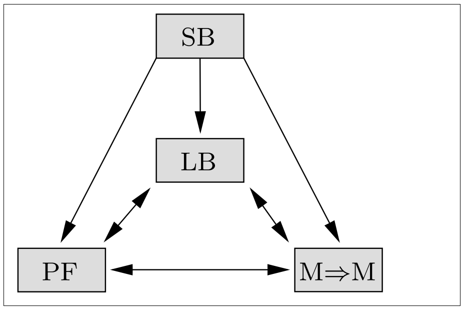
- It can be shown that for the following types of queueing systems the local balance property holds:
  - Type-1: **M/M/m--FCFS.** The service rates for different job classes must be equal.Examples of Type-1 nodes are input/output (I/O) devices or disks.
  - Type-2: **M/G/1--PS**. The CPU of a computer system can very often be modeled as a Type-2 node.
  - Type-3: **M/G/- IS** (infinite server)}. Terminals can be modeled as Type-3 nodes.
  - Type-4: **M/G/1--LCFS PR**. There is no practical example for the application of Type-4 nodes in computer systems.

2 Product-form Solutions

- A necessary and sufficient condition for the existence of product-form solutions is given in the **local balance property**:
  - The rate at which jobs enter a single node of the network is equated to the rate at which they leave it.
  - Thus local balance is concerned with a local situation and reduces the computational effort.

- Moreover, there exist two other characteristics that apply to a queueing network with **product-form solution**:
  - **M $\rightarrow$ M - Property** (Markov Implies Markov): A service station (node) has the M $\rightarrow$ M-property if and only if the station transforms a Poisson arrival process into a Poisson departure process. Muntz has shown that a queueing network has a product-form solution if all nodes of the network have the M $\rightarrow$ M-property.
  - **Station-balance property**

- Relation between local balance-, M $\rightarrow$ M-, station balance and der product-form property:



Jackson Networks:

- The networks examined fulfill the following assumptions:
  - There is only one job class in the network.
  - The overall number of jobs in the network is unlimited.
  - Each of the N nodes in the network can have arrivals from outside.
  - A job can leave the network from any node.
  - All service times and interarrival times are exponentially distributed.
  - The service discipline at all nodes is FCFS.
  - The *i*th node consists of $m_i > 1$ identical servers with the service rates μ_i. The arrival rates λ_{0i}, as well as the service rates, can depend on the number k_i of jobs at the node. In this case we have load-dependent service rates and load-dependent arrival rates.

- Theorem von Jackson:

If for all nodes $i = 1, \dots, N$ in the open network the condition for stability: $\lambda_i < m_i \cdot \mu_i$ is fulfilled, then the steady-state probability of the network can be expressed as the product of the state probabilities of the individual nodes, that is:

$$\pi(k_1, k_2, \dots, k_N) = \pi_1(k_1) \cdot \pi_2(k_2) \cdot \dots \cdot \pi_N(k_N)$$

- The nodes of the network can be considered as independent M/M/m-nodes with arrival rates λ_i and service rates μ_i. The arrival rates can be calculated using the traffic equations:

$$\lambda_i = \lambda_{0i} + \sum_{j=1}^{N} \lambda_j p_{ji}, \quad \text{for } i = 1, \dots, N$$

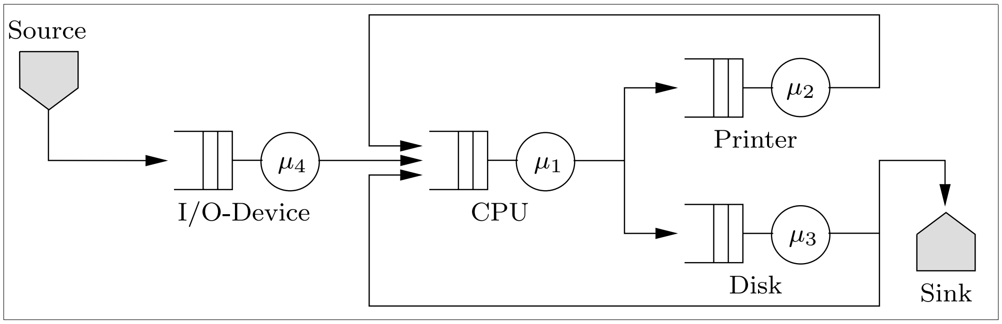
◆ Algorithm based on der Jackson's theorem:

- ***Step*** 1: For all nodes, $i = 1, \ldots, N$, compute the arrival rates λ_i of the open network by solving the traffic equations.

- ***Step*** 2: Consider each node as an M/M/m queueing system. Check the stability, and compute the state probabilities and performance measures of each node using the formulae for M/M/m-systems

- ***Step*** 3: Compute the steady-state probabilities of the overall network.

◆ Example:



- ➤ Number of nodes $N = 4$
- ➤ Service times exp. distr.with: $1/\mu_1 = 0.04$, $1/\mu_2 = 0.03$, $1/\mu_3 = 0.06$, $1/\mu_4 = 0.05$
- ➤ Interarrival times exp.distr. with: $\lambda = \lambda_{04} = 4$ jobs/sec
- ➤ Strategy: FCFS
- ➤ Transition probabilities:

$$p_{12} = p_{13} = 0.5, \quad p_{41} = p_{21} = 1, \quad p_{31} = 0.6, \quad p_{30} = 0.4$$

- ***Step*** 1: Arrival rates:

$$\lambda_1 = \lambda_2 p_{21} + \lambda_3 p_{31} + \lambda_4 p_{41} = \underline{20}\,, \quad \lambda_2 = \lambda_1 p_{12} = \underline{10}\,,$$
$$\lambda_3 = \lambda_1 p_{13} = \underline{10}\,, \quad \lambda_4 = \lambda_{04} = \underline{4}\,.$$

- ***Step*** 2: Performance measures:

➤ Utilizations:

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \underline{0.8}\,, \quad \rho_2 = \frac{\lambda_2}{\mu_2} = \underline{0.3}\,, \quad \rho_3 = \frac{\lambda_3}{\mu_3} = \underline{0.6}\,, \quad \rho_4 = \frac{\lambda_4}{\mu_4} = \underline{0.}$$

➤ Stability condition $\rho_i < 1$ is fulfilled for all nodes

➤ Mean number of jobs in the nodes:

$$\overline{K}_1 = \frac{\rho_1}{1-\rho_1} = \underline{4}\,, \quad \overline{K}_2 = \underline{0.429}\,, \quad \overline{K}_3 = \underline{1.5}\,, \quad \overline{K}_4 = \underline{0.25}$$

➤ Mean response times:

$$\overline{T}_1 = \frac{1/\mu_1}{1-\rho_1} = \underline{0.2}\,, \quad \overline{T}_2 = \underline{0.043}\,, \quad \overline{T}_3 = \underline{0.15}\,, \quad \overline{T}_4 = \underline{0.0625}\,.$$

➤ Mean overall response time (Little's law):

$$\overline{T} = \frac{\overline{K}}{\lambda} = \frac{1}{\lambda}\sum_{1=1}^{4} \overline{K}_i = \underline{1.545}\,.$$

➤ Mean waiting time:

$$\overline{W}_1 = \frac{\rho_1/\mu_1}{1-\rho_1} = \underline{0.16}\,, \quad \overline{W}_2 = \underline{0.013}\,, \quad \overline{W}_3 = \underline{0.09}\,, \quad \overline{W}_4 = \underline{0.0125}$$

➤ Mean queue length:

$$\overline{Q}_1 = \frac{\rho_1^2}{1-\rho_1} = \underline{3.2}\,, \quad \overline{Q}_2 = \underline{0.129}\,, \quad \overline{Q}_3 = \underline{0.9}\,, \quad \overline{Q}_4 = \underline{0.05}$$

➤ Marginal probabilities:

$$\pi_1(3) = (1-\rho_1)\rho_1^3 = \underline{0.1024}\,, \quad \pi_2(2) = (1-\rho_2)\rho_2^2 = \underline{0.063}\,,$$
$$\pi_3(4) = (1-\rho_3)\rho_3^4 = \underline{0.0518}\,, \quad \pi_4(1) = (1-\rho_4)\rho_4 = \underline{0.16}\,.$$

- ***Step*** 3: Steady state probability of the network:

$$\pi(3,2,4,1) = \pi_1(3)\cdot\pi_2(2)\cdot\pi_3(4)\cdot\pi_4(1) = \underline{0.0000534}$$

Gordon/Newell-Theorem for Closed Networks:

◆ For closed networks with the same assumptions as for Jackson's theorem for open queueing networks, except that no job can enter or leave the system ($\lambda_{0i} = \lambda_{i0} = 0$) and:

$$K = \sum_{i=1}^{N} k_i$$

◆ Thus, the number of possible states is finite, and it is given by the binomial coefficient:

$$\binom{N+K-1}{N-1}$$

◆ Gordon/Newell-Theorem:

Steady state probabilities:

$$\pi(k_1, \ldots, k_N) = \frac{1}{G(K)} \prod_{i=1}^{N} F_i(k_i)$$

➤ Normalizing constant:

$$G(K) = \sum_{\sum_{i=1}^{N} k_i = K} \prod_{i=1}^{N} F_i(k_i)$$

➤ Functions $F_i\,(k_i)$:

$$F_i(k_i) = \left(\frac{e_i}{\mu_i}\right)^{k_i} \cdot \frac{1}{\beta_i(k_i)}$$

with:

$$\beta_i(k_i) = \begin{cases} k_i!\,, & k_i \le m_i\,, \\ m_i! \cdot m_i^{k_i - m_i}\,, & k_i \ge m_i\,, \\ 1\,, & m_i = 1\,. \end{cases}$$

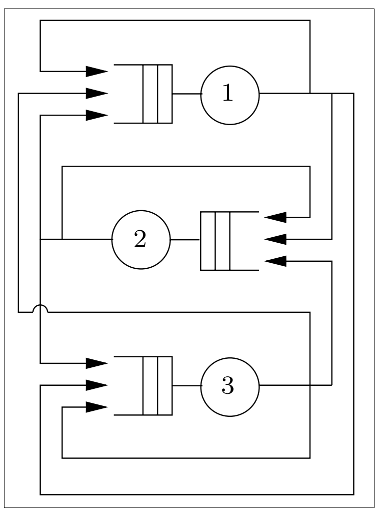
and:

$$e_i = \sum_{j=1}^{N} e_j p_{ji}\,, \quad \text{for } i = 1, \ldots, N$$

◆ Gordon/Newell-Method:

- ***Step*** 1: Compute the visit ratios e_i for all nodes i.
- ***Step*** 2: Compute the functions $F_i(k_i)$ for all nodes i = 1, ... , N.
- ***Step*** 3: Compute the normalization constant $G(K)$.
- ***Step*** 4: Compute the state probabilities of the network .
- ***Step*** 5: Compute the marginal probabilities from the state probabilities
- ***Step*** 6: Compute all other required performance measures from the marginal probabilities

◆ Example:



➤ Number of Jobs: $K = 3$

➤ Mean service times (exp. distr.) with $\mu_1 = 0.8/\text{sec}$, $\mu_2 = 0.6/\text{sec}$, $\mu_3 = 0.4/\text{sec}$

➤ Strategy: FCFS

➤ Transition probabilities:

$$p_{11} = 0.6\,, \quad p_{21} = 0.2\,, \quad p_{31} = 0.4\,,$$
$$p_{12} = 0.3\,, \quad p_{22} = 0.3\,, \quad p_{32} = 0.1\,,$$
$$p_{13} = 0.1\,, \quad p_{23} = 0.5\,, \quad p_{33} = 0.5\,.$$

➤ Number of states:

$$\binom{N+K-1}{N-1} = 10$$

➤ States:

$$(3,0,0)\,, \quad (2,1,0)\,, \quad (2,0,1)\,, \quad (1,2,0)\,, \quad (1,1,1)\,,$$
$$(1,0,2)\,, \quad (0,3,0)\,, \quad (0,2,1)\,, \quad (0,1,2)\,, \quad (0,0,3)\,.$$

• ***Step*** 1: Visit ratios:

$$e_1 = e_1 p_{11} + e_2 p_{21} + e_3 p_{31} = \underline{1}\,,$$
$$e_2 = e_1 p_{12} + e_2 p_{22} + e_3 p_{32} = \underline{0.533}\,,$$
$$e_3 = e_1 p_{13} + e_2 p_{23} + e_3 p_{33} = \underline{0.733}\,.$$

• ***Step*** 2: Functions $F_i(k_i)$:

$$F_1(0) = (e_1/\mu_1)^0 = \underline{1}\,, \qquad F_1(1) = (e_1/\mu_1)^1 = \underline{1.25}\,,$$
$$F_1(2) = (e_1/\mu_1)^2 = \underline{1.5625}\,, \quad F_1(3) = (e_1/\mu_1)^3 = \underline{1.953}\,,$$

Respectively:

$$F_2(0) = \underline{1}\,, \quad F_2(1) = \underline{0.889}\,, \quad F_2(2) = \underline{0.790}\,, \quad F_2(3) = \underline{0.702}\,,$$
$$F_3(0) = \underline{1}\,, \quad F_3(1) = \underline{1.833}\,, \quad F_3(2) = \underline{3.361}\,, \quad F_3(3) = \underline{6.162}\,.$$

- ***Step*** 3: Normalization constant:

$$\begin{aligned} G(3) = \; & F_1(3)F_2(0)F_3(0) + F_1(2)F_2(1)F_3(0) + F_1(2)F_2(0)F_3(1) \\ & + F_1(1)F_2(2)F_3(0) + F_1(1)F_2(1)F_3(1) + F_1(1)F_2(0)F_3(2) \\ & + F_1(0)F_2(3)F_3(0) + F_1(0)F_2(2)F_3(1) + F_1(0)F_2(1)F_3(2) \\ & + F_1(0)F_2(0)F_3(3) = \underline{24.733}\,. \end{aligned}$$

- ***Step*** 4: State probabilities:

$$\pi(3,0,0) = \frac{1}{G(3)} F_1(3)\cdot F_2(0)\cdot F_3(0) = \underline{0.079}\,,$$
$$\pi(2,1,0) = \frac{1}{G(3)} F_1(2)\cdot F_2(1)\cdot F_3(0) = \underline{0.056}\,.$$

Respectively:

$$\pi(2,0,1) = \underline{0.116}\,,\quad \pi(1,2,0) = \underline{0.040}\,,\quad \pi(1,1,1) = \underline{0.082}\,,\quad \pi(1,0,2) = \underline{0.170}\,,$$
$$\pi(0,3,0) = \underline{0.028}\,,\quad \pi(0,2,1) = \underline{0.058}\,,\quad \pi(0,1,2) = \underline{0.121}\,,\quad \pi(0,0,3) = \underline{0.249}\,.$$

- ***Step*** 5: Marginal probabilities:

$$\pi_1(0) = \pi(0,3,0) + \pi(0,2,1) + \pi(0,1,2) + \pi(0,0,3) = \underline{0.457}\,,$$
$$\pi_1(1) = \pi(1,2,0) + \pi(1,1,1) + \pi(1,0,2) = \underline{0.292}\,,$$
$$\pi_1(2) = \pi(2,1,0) + \pi(2,0,1) = \underline{0.172}\,,$$
$$\pi_1(3) = \pi(3,0,0) = \underline{0.079}\,,$$

$$\pi_2(0) = \pi(2,0,1) + \pi(1,0,2) + \pi(0,0,3) + \pi(3,0,0) = \underline{0.614}\,,$$
$$\pi_2(1) = \pi(2,1,0) + \pi(1,1,1) + \pi(0,1,2) = \underline{0.259}\,,$$
$$\pi_2(2) = \pi(1,2,0) + \pi(0,2,1) = \underline{0.098}\,,$$
$$\pi_2(3) = \pi(0,3,0) = \underline{0.028}\,,$$

$$\pi_3(0) = \pi(3,0,0) + \pi(2,1,0) + \pi(1,2,0) + \pi(0,3,0) = \underline{0.203}\,,$$
$$\pi_3(1) = \pi(2,0,1) + \pi(1,1,1) + \pi(0,2,1) = \underline{0.257}\,,$$
$$\pi_3(2) = \pi(1,0,2) + \pi(0,1,2) = \underline{0.291}\,,$$
$$\pi_3(3) = \pi(0,0,3) = \underline{0.249}\,.$$

- ***Step*** 6: Performance measures:
  - ➤ Utilization:

$$\rho_1 = 1 - \pi_1(0) = \underline{0.543}\,, \quad \rho_2 = \underline{0.386}\,, \quad \rho_3 = \underline{0.797}$$

  - ➤ Mean number of jobs:

$$\overline{K}_1 = \sum_{k=1}^{3} k \cdot \pi_1(k) = \underline{0.873}\,, \quad \overline{K}_2 = \underline{0.541}, \quad \overline{K}_3 = \underline{1.585}$$

  - ➤ Throughputs:

$$\lambda_1 = m_1 \rho_1 \mu_1 = \underline{0.435}\,, \quad \lambda_2 = \underline{0.232}\,, \quad \lambda_3 = \underline{0.319}$$

  - ➤ Response times:

$$\overline{T}_1 = \frac{\overline{K}_1}{\lambda_1} = \underline{2.009}\,, \quad \overline{T}_2 = \underline{2.337}\,, \quad \overline{T}_3 = \underline{4.976}$$

BCMP-Networks (BCMP-Theorem):

◆ **Conditions**:

➤ Node types:

Type-1:	M/M/m – FCFS	Type-2:	M/G/1 – PS
Type-3:	$M/G/\infty$ (IS)	Type-4:	M/G/1 – LCFS PR

➤ Multiple job classes

➤ For the strategy FCFS the service rates have to be identical

➤ For the strategies IS, PS and LCFS-PR the service rates must not be identical

➤ Service rates can be load dependent (dependent on the number of jobs in the node)

➤ Open, closed and mixed networks

➤ Theorem:

$$\pi(\mathrm{S}_1, \ldots, \mathrm{S}_N) = \frac{1}{G(\mathbf{K})} d(\mathrm{S}) \prod_{i=1}^{N} f_i(\mathrm{S}_i)$$

◆ **Closed queueing networks with multiple job classes:**

➤ State probabilities:

$$\pi(\mathbf{S}_1, \ldots, \mathbf{S}_N) = \frac{1}{G(\mathbf{K})} \prod_{i=1}^{N} F_i(\mathbf{S}_i)$$

➤ Normalization constant:

$$G(\mathbf{K}) = \sum_{\sum_{i=1}^{N} \mathbf{S}_i = \mathbf{K}} \prod_{i=1}^{N} F_i(\mathbf{S}_i)$$

➤ Functions $F_i(\mathbf{S}_i)$:

$$F_i(\mathbf{S}_i) = \begin{cases} k_i! \dfrac{1}{\beta_i(k_i)} \cdot \left(\dfrac{1}{\mu_i}\right)^{k_i} \cdot \prod_{r=1}^{R} \dfrac{1}{k_{ir}!} e_{ir}^{k_{ir}}, & \text{Type-1,} \\ k_i! \prod_{r=1}^{R} \dfrac{1}{k_{ir}!} \cdot \left(\dfrac{e_{ir}}{\mu_{ir}}\right)^{k_{ir}}, & \text{Type-2,4,} \\ \prod_{r=1}^{R} \dfrac{1}{k_{ir}!} \cdot \left(\dfrac{e_{ir}}{\mu_{ir}}\right)^{k_{ir}}, & \text{Type-3.} \end{cases}$$

with:

$$k_i = \sum_{r=1}^{R} k_{ir}$$

◆ **Open queueing networks with multiple job classes:**

◆ State probabilities:

$$\pi(k_1, \ldots, k_N) = \prod_{i=1}^{N} \pi_i(k_i)$$

➤ Marginal probabilities:

$$\pi_i(k_i) = \begin{cases} (1-\rho_i)\rho_i^{k_i}, & \text{Type-1,2,4 } (m_i = 1), \\ e^{-\rho_i} \dfrac{\rho_i^{k_i}}{k_i!}, & \text{Type-3,} \end{cases}$$

with:

$$k_i = \sum_{i=i}^{R} k_{ir}$$

➤ Utilizations

$$\rho_i = \sum_{r=1}^{R} \rho_{ir}$$

$$\rho_{ir} = \begin{cases} \lambda_r \dfrac{e_{ir}}{\mu_i}, & \text{Type-1 } (m_i = 1), \\ \lambda_r \dfrac{e_{ir}}{\mu_{ir}}, & \text{Type-2,3,4.} \end{cases}$$

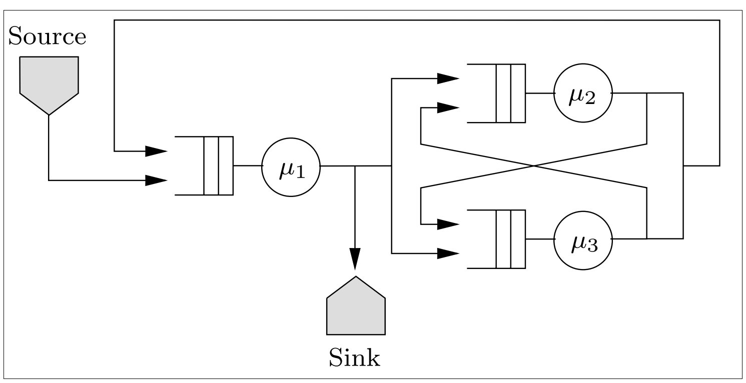
➤ Mean number of jobs:

$$\overline{K}_{ir} = \frac{\rho_{ir}}{1-\rho_i}$$

◆ Algorithm for open networks with multiple job classes using the BCMP-theorem:

- ***Step*** 1: Compute the visit ratios e_{ir} for all nodes i
- ***Step*** 2: Compute the utilization of each node ρ_{ir} and ρ_i of each node
- ***Step*** 3: Compute the mean number of jobs $\overline{K}_{ir}$ and other performance measures
- ***Step*** 4: Compute the marginal probabilities of the network
- ***Step*** 5: Compute the state probabilities

◆ Example



➤ Number of nodes $N = 3$
➤ Number of job classes $R = 2$
➤ Node 1: Type-2
➤ Node 2 and 3: Type-4

➤ Service times (exp. distributed) with the rates:

$$\mu_{11} = 8\,\text{sec}^{-1}\,,\quad \mu_{21} = 12\,\text{sec}^{-1}\,,\quad \mu_{31} = 16\,\text{sec}^{-1}\,,$$
$$\mu_{12} = 24\,\text{sec}^{-1}\,,\quad \mu_{22} = 32\,\text{sec}^{-1}\,,\quad \mu_{32} = 36\,\text{sec}^{-1}\,.$$

➤ Interarrival times (exp. distributed) with the rates:

$$\lambda_1 = \lambda_2 = 1 \quad \text{Auftraege/sec}$$

➤ Transition probabilities:

$$\begin{array}{llll} p_{0,11} = 1\,, & p_{21,11} = 0.6\,, & p_{0,12} = 1\,, & p_{22,12} = 0.7\,, \\ p_{11,21} = 0.4\,, & p_{21,31} = 0.4\,, & p_{12,22} = 0.3\,, & p_{22,32} = 0.3\,, \\ p_{11,31} = 0.3\,, & p_{31,11} = 0.5\,, & p_{12,32} = 0.6\,, & p_{32,12} = 0.4\,, \\ p_{11,0} = 0.3\,, & p_{31,21} = 0.5\,, & p_{12,0} = 0.1\,, & p_{32,22} = 0.6\,, \end{array}$$

- ***Step*** 1: Visit ratios:

$$e_{11} = p_{0,11} + e_{11}p_{11,11} + e_{21}p_{21,11} + e_{31}p_{31,11} = \underline{3.333}\,,$$
$$e_{21} = p_{0,21} + e_{11}p_{11,21} + e_{21}p_{21,21} + e_{31}p_{31,21} = \underline{2.292}\,,$$
$$e_{31} = p_{0,31} + e_{11}p_{11,31} + e_{21}p_{21,31} + e_{31}p_{31,31} = \underline{1.917}\,.$$

and

$$e_{12} = \underline{10},\quad e_{22} = \underline{8.049},\quad e_{32} = \underline{8.415}$$

- ***Step*** 2: Utilizations:

$$\rho_1 = \lambda_1\frac{e_{11}}{\mu_{11}} + \lambda_2\frac{e_{12}}{\mu_{12}} = \rho_{11} + \rho_{12} = \underline{0.833}\,,$$
$$\rho_2 = \lambda_1\frac{e_{21}}{\mu_{21}} + \lambda_2\frac{e_{22}}{\mu_{22}} = \rho_{21} + \rho_{22} = \underline{0.442}\,,$$
$$\rho_3 = \lambda_1\frac{e_{31}}{\mu_{31}} + \lambda_2\frac{e_{32}}{\mu_{32}} = \rho_{31} + \rho_{32} = \underline{0.354}\,.$$

- ***Step*** 3: Mean number of jobs:

$$\overline{K}_{11} = \frac{\rho_{11}}{1-\rho_1} = \underline{2.5}, \quad \overline{K}_{21} = \frac{\rho_{21}}{1-\rho_2} = \underline{0.342}, \quad \overline{K}_{31} = \frac{\rho_{31}}{1-\rho_3} = \underline{0.186},$$

$$\overline{K}_{12} = \underline{2.5}, \quad \overline{K}_{22} = \underline{0.5}, \quad \overline{K}_{32} = \underline{0.362}.$$

- ***Step*** 4: Marginal probabilities:

$$\pi_1(3) = (1-\rho_1)\rho_1^3 = \underline{0.0965}, \quad \pi_2(2) = (1-\rho_2)\rho_2^2 = \underline{0.1093},$$
$$\pi_3(1) = (1-\rho_3)\rho_3 = \underline{0.2287}.$$

- ***Step*** 5: State probabilities:

$$\pi(3,2,1) = \pi_1(3)\cdot\pi_2(2)\cdot\pi_3(1) = \underline{0.00241}$$