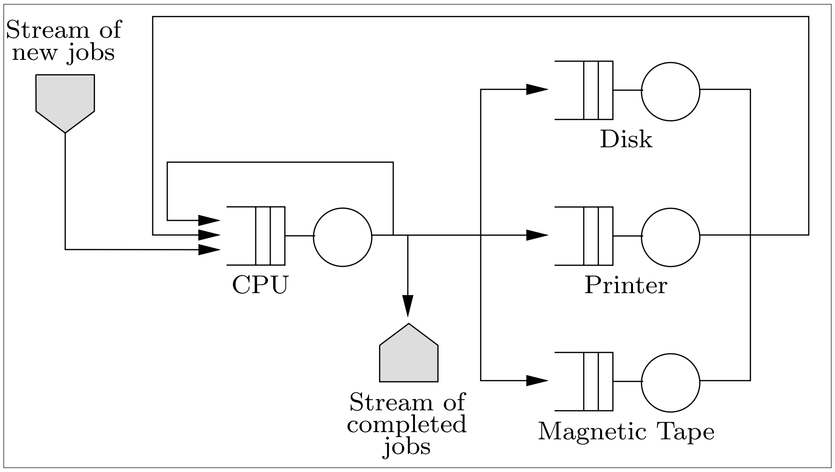
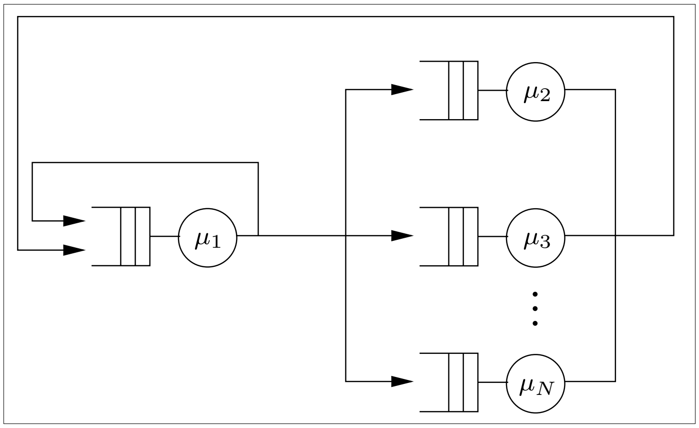
E Queueing Networks

◆ **Open** queueing network model of a computer system:



◆ "**Central-Server-Model** (CSM)" : Computer system modelled as a **closed** queueing network



E.1 Description

- Notation

N	Number of nodes
K	The constant number of jobs in a closed network
$(k_1, k_2, \ldots, k_N)$	The state of the network
k_i	The number of jobs at the ith node; for closed networks $\sum_{i=1}^{N} k_i = K$
m_i	The number of parallel servers at the ith node $(m_i \geq 1)$
μ_i	Service rate of the jobs at the ith node
$1/\mu_i$	The mean service time of the jobs at the ith node

p_{ij}	Routing probability, the probability that a job is transferred to the jth node after service completion at the ith node. (In open networks, the node with index 0 represents the external world to the network.)
p_{0j}	The probability that a job entering the network from outside first enters the jth node
p_{i0}	The probability that a job leaves the network just after completing service at node i $(p_{i0} = 1 - \sum_{j=1}^{N} p_{ij})$.
λ_{0i}	The arrival rate of jobs from outside to the ith node
λ	The overall arrival rate from outside to an open network $(\lambda = \sum_{i=1}^{N} \lambda_{0i})$
λ_i	the overall arrival rate of jobs at the ith node

- Fundamental formulas:

  ◆ **Traffic equations** (open networks):

$$\lambda_i = \lambda_{0i} + \sum_{j=1}^{N} \lambda_j p_{ji}\,, \quad \text{for } i = 1, \ldots, N$$

  ➤ **Traffic equations** (closed networks):

$$\lambda_i = \sum_{j=1}^{N} \lambda_j p_{ji}\,, \quad \text{for } i = 1, \ldots, N$$

◆ **Visit ratio** (mean number of visits, relative arrival rate):

$$e_i = \frac{\lambda_i}{\lambda}\,, \quad \text{for } i = 1, \ldots, N$$

λ: Overall throughput of the network

➤ Open networks:

$$e_i = p_{0i} + \sum_{j=1}^{N} e_j p_{ji}\,, \quad \text{for } i = 1, \ldots, N$$

➤ Closed networks:

$$e_i = \sum_{j=1}^{N} e_j p_{ji}\,, \quad \text{for } i = 1, \ldots, N$$

E.2 Performance Measures

- **Steady-state probabilies**:

$$\pi(k_1, \ldots, k_N)$$

Probability, network is in state $(k_1, \ldots, k_N)$

- **Marginal probability** (P(k jobs in node i)):
  - Open networks:

$$\pi_i(k) = \sum_{k_i = k} \pi(k_1, \ldots, k_N)$$

Normalizing condition:

$$\sum \pi(k_1, \ldots, k_N) = 1$$

- Closed networks (number of jobs is constant):

$$\sum_{j=1}^{N} k_j = K \text{ with } (0 \le k_j \le K)$$

$$\pi_i(k) = \sum_{\substack{\sum_{j=1}^{N} k_j = K \\ \& \; k_i = k}} \pi(k_1, \ldots, k_N)$$

Normalizing condition:

$$\sum_{\sum_{j=1}^{N} k_j = K} \pi(k_1, \ldots, k_N) = 1$$

◆ **Utilization** of node i:

➤ $m_i = 1$:

$$\rho_i = \sum_{k=1}^{\infty} \pi_i(k)$$

or:

$$\rho_i = 1 - \pi_i(0)$$

➤ $m_i > 1$:

$$\rho_i = \frac{1}{m_i} \sum_{k=0}^{\infty} \min(m_i, k) \pi_i(k) = 1 - \sum_{k=0}^{m_i - 1} \frac{m_i - k}{m_i} \cdot \pi_i(k)$$

or:

$$\rho_i = \frac{\lambda_i}{m_i \mu_i}$$

◆ **Throughput** of node i:

$$\lambda_i = m_i \cdot \rho_i \cdot \mu_i$$

➤ Load dependent node (Service rate depends on the number of jobs in the node):

$$\lambda_i = \sum_{k=1}^{\infty} \pi_i(k) \mu_i(k)$$

Example for a load dependent node → M/M/m-node:

$$\mu_i(k) = \min(k, m_i) \cdot \mu_i$$

μ_i: Service rate of a single server

◆ **Overall throughput**:

➤ Open network:

$$\lambda = \sum_{i=1}^{N} \lambda_{0i}$$

➤ Closed network:

$$\lambda = \frac{\lambda_i}{e_i}$$

◆ Mean **number of jobs** in node i:

$$\overline{K}_i = \sum_{k=1}^{\infty} k \cdot \pi_i(k)$$

Little’s law:

$$\overline{K}_i = \lambda_i \cdot \overline{T}_i$$

◆ Mean **queue length** of node i:

$$\overline{Q}_i = \sum_{k=m_i}^{\infty} (k - m_i) \cdot \pi_i(k)$$

Little:

$$\overline{Q}_i = \lambda_i \overline{W}_i$$

◆ Mean **response time** of node i (Little):

$$\overline{T}_i = \frac{\overline{K}_i}{\lambda_i}$$

◆ Mean **overall response time** (Little):

$$\overline{T} = \frac{\overline{K}}{\lambda}$$

◆ Mean **waiting time** of node i:

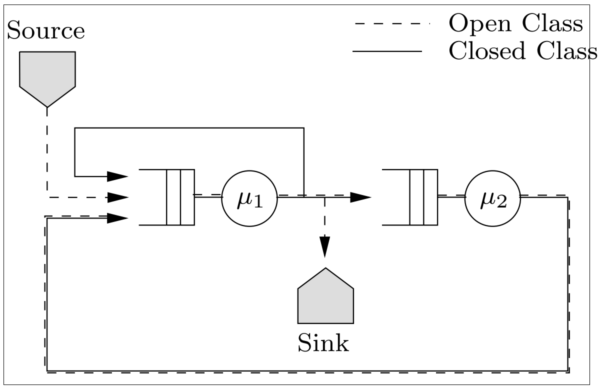
$$\overline{W}_i = \overline{T}_i - \frac{1}{\mu_i}$$

E.3 Queueing Networks with multiple Job Classes

➤ Multiple job classes in the network

➤ Jobs of different classes have different **service rates** and different **routing probabilities** (transition probabilities)

➤ Networks with open and closed classes are called **mixed networks:**



1 Notation

R The number of job classes in a network

k_{ir} The number of jobs of the rth class at the ith node; for a closed network:

$$\sum_{i=1}^{N} \sum_{r=1}^{R} k_{ir} = K \tag{0.1}$$

K_r The number of jobs of the rth class in the network; not necessarily constant, even in a closed network:

$$\sum_{i=1}^{N} k_{ir} = K_r \tag{0.2}$$

$\mathbf{K}$ The number of jobs in the various classes, known as the population vector ($\mathbf{K} = (K_1, \ldots, K_R)$)

$\mathbf{S}_i$ The state of the ith node ($\mathbf{S}_i = (k_{i1}, \ldots, k_{iR})$):

$$\sum_{i=1}^{N} \mathbf{S}_i = \mathbf{K} \tag{0.1}$$

$\mathbf{S}$ The overall state of the network with multiple classes ($\mathbf{S} = (\mathbf{S}_1, \ldots, \mathbf{S}_N)$)

μ_{ir} The service rate of the ith node for jobs of the rth class

$p_{ir,js}$ The probability that a job of the rth class at the ith node is transferred to the sth class and the jth node (routing probability)

$p_{0,js}$ The probability in an open network that a job from outside the network enters the jth node as a job of the sth class

$p_{ir,0}$ The probability in an open network that a job of the rth class leaves the network after having been serviced at the ith node, so:

$$p_{ir,0} = 1 - \sum_{j=1}^{N} \sum_{s=1}^{R} p_{ir,js} \tag{0.2}$$

λ The overall arrival rate from outside to an open network

$\lambda_{0,ir}$ The arrival rate from outside to node i for class r jobs ($\lambda_{0,ir} = \lambda \cdot p_{0,ir}$)

λ_{ir} The arrival rate of jobs of the rth class at the ith node:

$$\lambda_{ir} = \lambda \cdot p_{0,ir} + \sum_{j=1}^{N} \sum_{s=1}^{R} \lambda_{js} \cdot p_{js,ir} \, ; \tag{0.1}$$

for closed networks, $p_{0,ir} = 0$ $(1 < i < N\,, 1 < r < R)$ and we obtain:

$$\lambda_{ir} = \sum_{j=1}^{N} \sum_{s=1}^{R} \lambda_{js} \cdot p_{js,ir} \, . \tag{0.2}$$

The mean number of visits e_{ir} of a job of the rth class at the ith node of an open network can be determined from the routing probabilities:

$$e_{ir} = p_{0,ir} + \sum_{j=1}^{N} \sum_{s=1}^{R} e_{js} p_{js,ir} \, , \quad \text{for } i = 1, \ldots, N\,, \; r = 1, \ldots, R\,. \tag{0.1}$$

For closed networks, the corresponding equation is:

$$e_{ir} = \sum_{j=1}^{N} \sum_{s=1}^{R} e_{js} p_{js,ir} \, , \qquad \text{for } i = 1, \ldots, N\,, \; r = 1, \ldots, R\,. \tag{0.2}$$

Usually we assume that $e_{1r} = 1$, for $r = 1, \ldots, R$, although other settings are also possible.

2 Performance Measures

◆ **Steady-state probability**:

$$\pi(S_1, \ldots, S_N)$$

Probability, network is in state $(S_1, \ldots, S_N)$

◆ **Marginal probability** (P(node *i* is in state ***k***) = P($\mathbf{S}_i$ = ***k***)):

➤ Open networks:

$$\pi_i(\mathbf{k}) = \sum_{s_i = k} \pi(S_1, \ldots, S_N)$$

➤ Closed networks:

$$\pi_i(\mathbf{k}) = \sum_{\substack{\sum_{j=1}^{N} \mathbf{S}_j = \mathbf{K} \\ \& \ \mathbf{S}_i = \mathbf{k}}} \pi(S_1, \ldots, S_N),$$

◆ **Utilization** of node *i* with jobs of class *r*:

$$\rho_{ir} = \frac{1}{m_i} \sum_{\substack{\text{all states } \mathbf{k} \\ \text{with } k_r > 0}} \pi_i(\mathbf{k}) \frac{k_{ir}}{k_i} \min(m_i, k_i), \quad k_i = \sum_{r=1}^{R} k_{ir}$$

or:

$$\rho_{ir} = \frac{\lambda_{ir}}{m_i \mu_{ir}}$$

◆ **Throughput** of node *i* with jobs of class *r*:

$$\lambda_{ir} = \sum_{\substack{\text{all states } \mathbf{k} \\ \text{with } k_r > 0}} \pi_i(\mathrm{k}) \frac{k_{ir}}{k_i} \mu_i(k_i)$$

or:

$$\lambda_{ir} = m_i \cdot \rho_{ir} \cdot \mu_{ir}$$

◆ **Overall throughput** of the network with jobs of class *r*:

➤ Open network:

$$\lambda_r = \sum_{i=1}^{N} \lambda_{0,ir}$$

➤ Closed network:

$$\lambda_r = \frac{\lambda_{ir}}{e_{ir}}$$

◆ Mean **number of jobs** of class *r* in node *i*:

$$\overline{K}_{ir} = \sum_{\substack{\text{all states } \mathbf{k} \\ \text{with } k_r > 0}} k_r \cdot \pi_i(\mathrm{k})$$

Little:

$$\overline{K}_{ir} = \lambda_{ir} \cdot \overline{T}_{ir}$$

◆ Mean **queue length** of jobs of class *r* in node *i*:

$$\overline{Q}_{ir} = \lambda_{ir} \overline{W}_{ir}$$

◆ Mean **response time** of jobs of class *r* in node *i*:

$$\overline{T}_{ir} = \frac{\overline{K}_{ir}}{\lambda_{ir}}$$

◆ Mean **waiting time** of jobs of class *r:*

$$\overline{W}_{ir} = \overline{T}_{ir} - \frac{1}{\mu_{ir}}$$