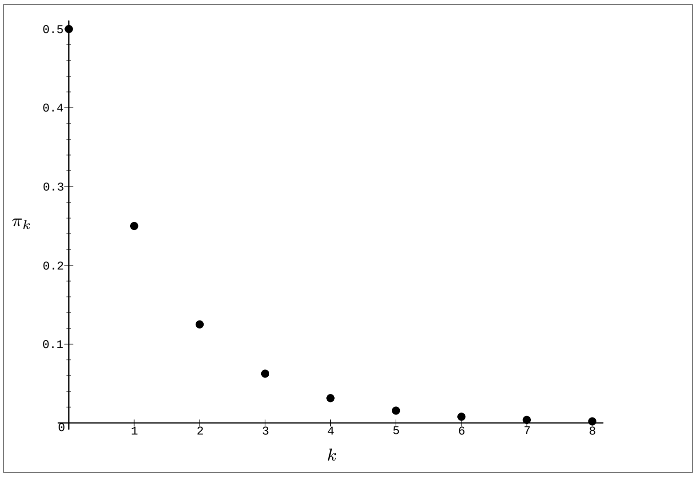
D.4 FIFO-Systems

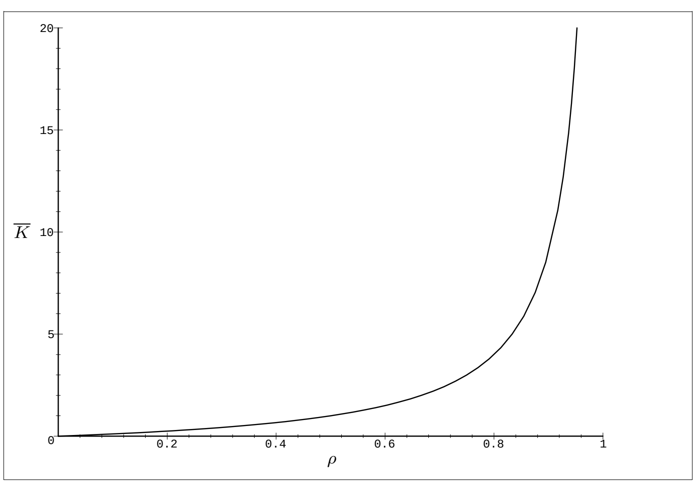
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}\frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

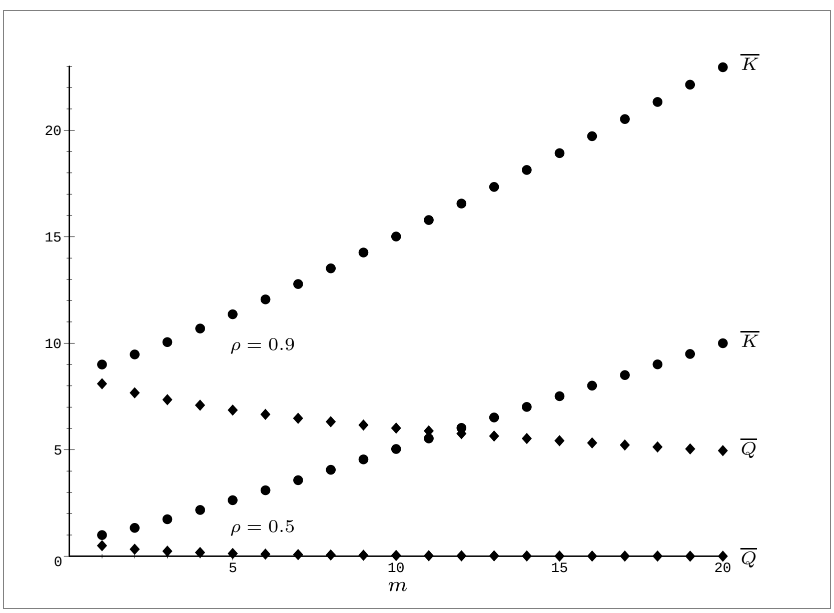
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \leq k \leq K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

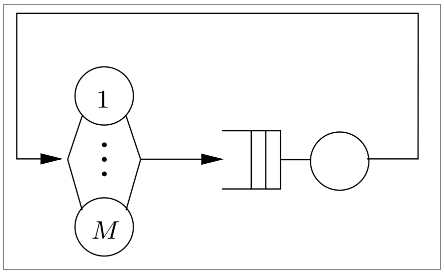
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

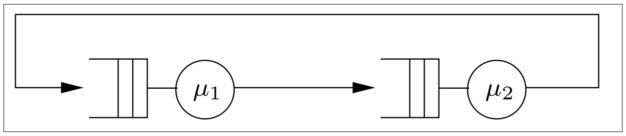
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\[2ex] \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} ,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} , & u \neq 1 , \\ \dfrac{1}{K+1} , & u = 1 . \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) , \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1 \mu_1 = \rho_2 \mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} , \quad u \neq 1 ,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little’s law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

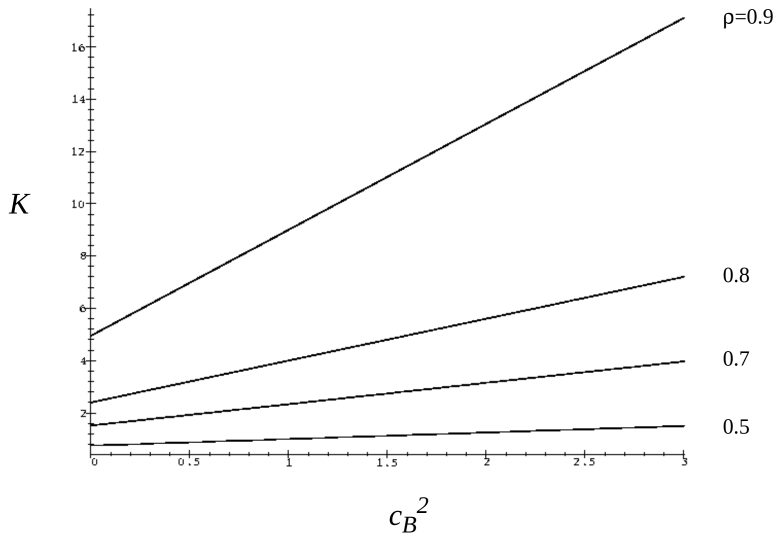
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

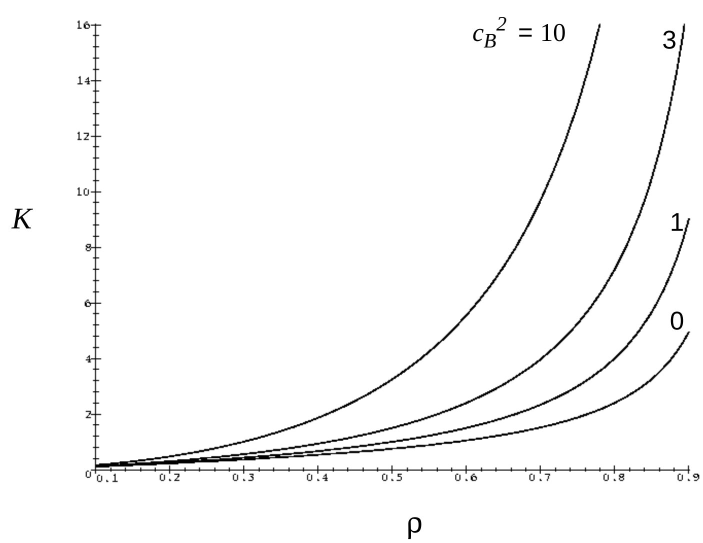
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

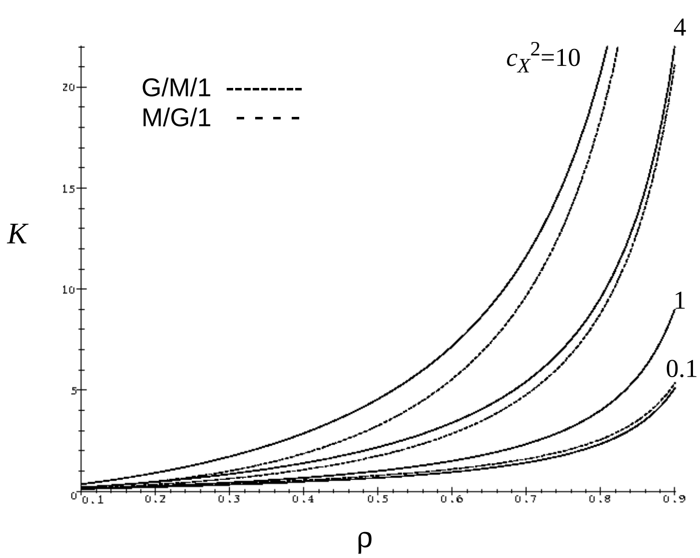
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

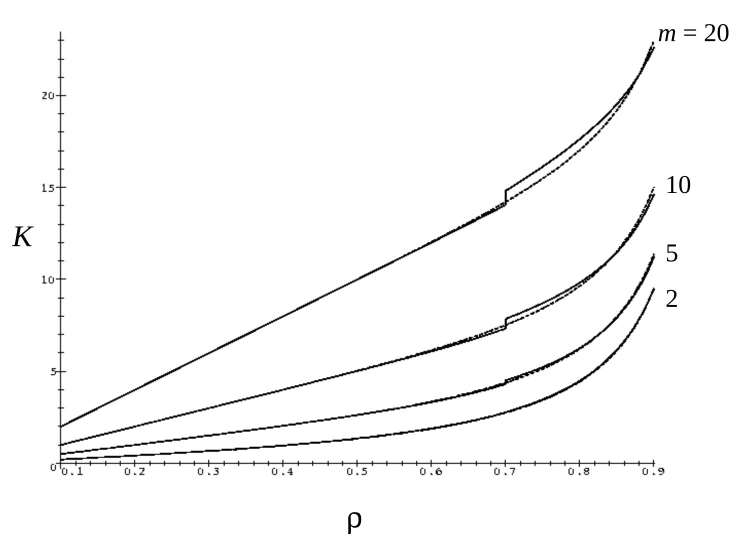
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

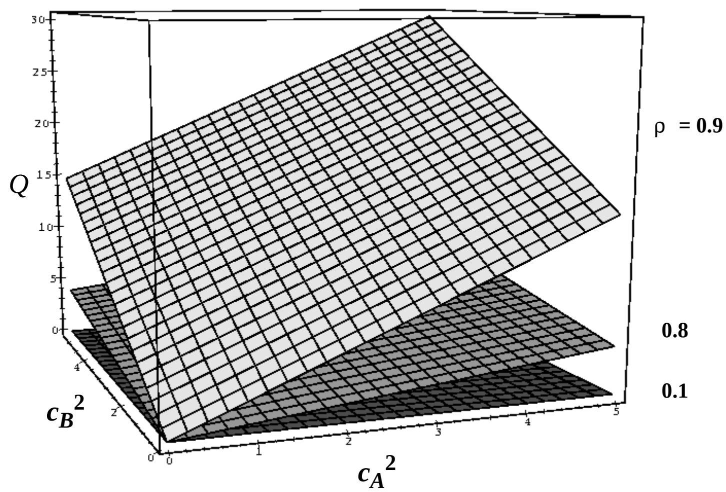
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

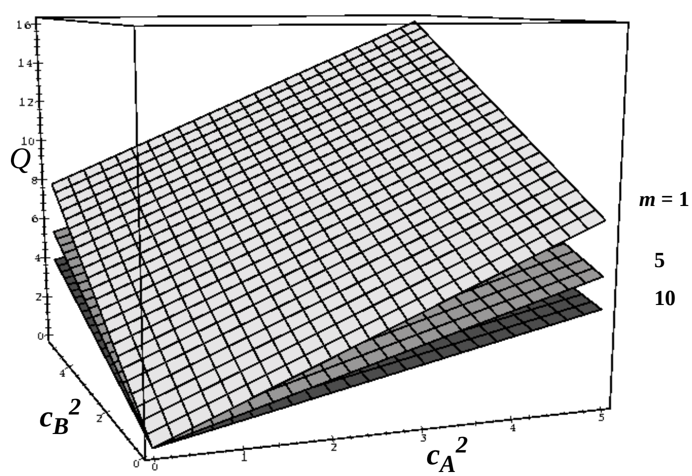
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\dfrac{2}{3}\dfrac{1-\rho}{P_m}\dfrac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\dfrac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

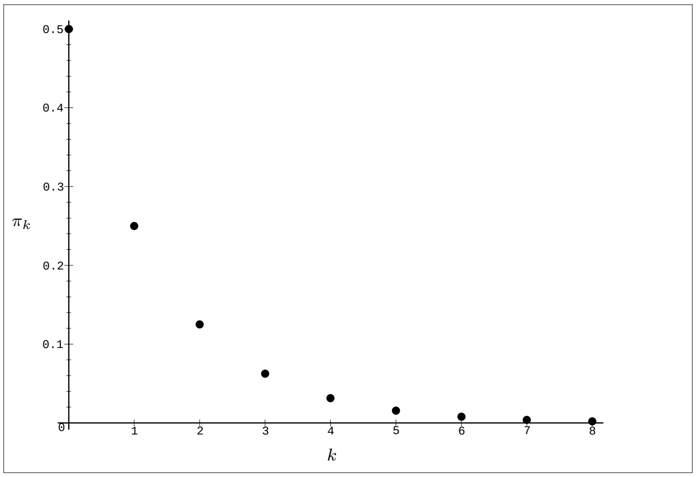


• Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

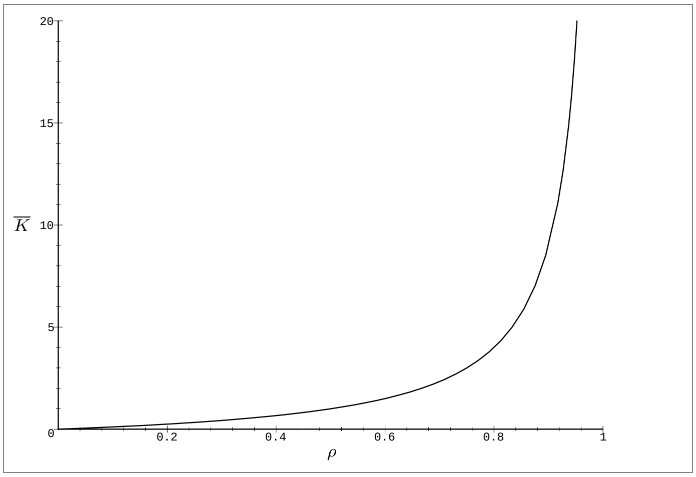
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho} \right]^{-1}$$

- Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

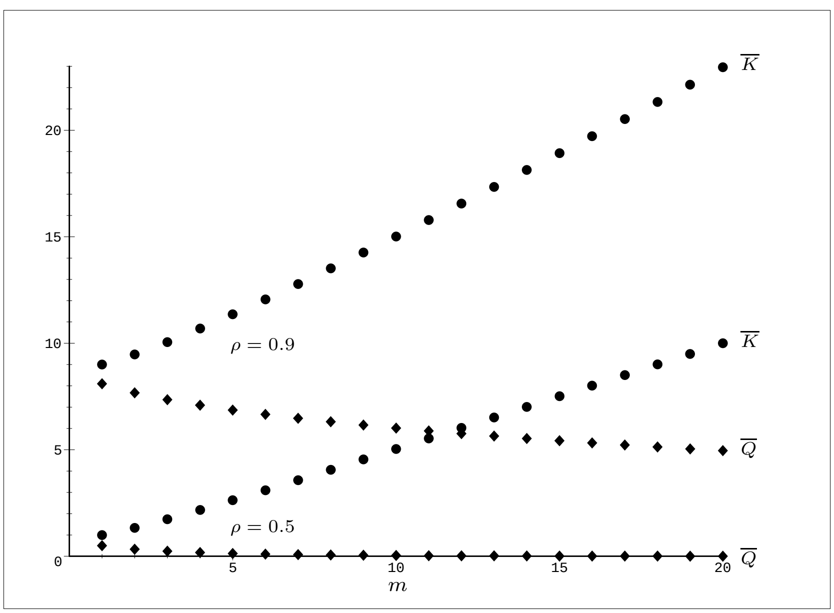
- Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

- Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

- Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

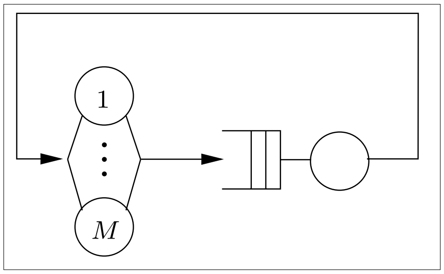
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



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➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

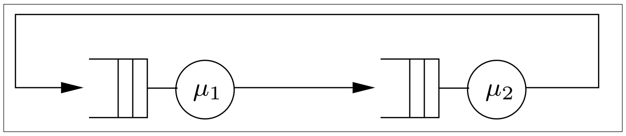
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P(k_1$ *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} ,$$
$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} , & u \neq 1 , \\ \dfrac{1}{K+1} , & u = 1 . \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) , \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} , \quad u \neq 1 ,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

◆ System is given by:

➤ Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$

➤ Service time arbitrarily distributed:

➤ Mean: $\overline{T}_B = 1/\mu$

➤ Coefficient of variation: c_B

➤ Number of servers: $m = 1$

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➤ The mean remaining service time $\overline{W}_0$ of ther job in service (if any).

➤ The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1 - \rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{\boldsymbol{R}} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System ($c_B = 1$) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little’s law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

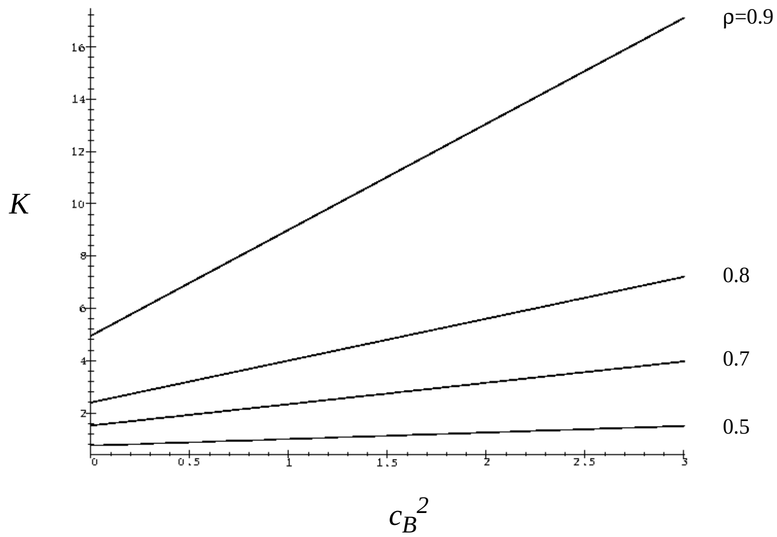
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems ($c_B = 1$) and M/D/1-systems ($c_B = 0$) we obtain:

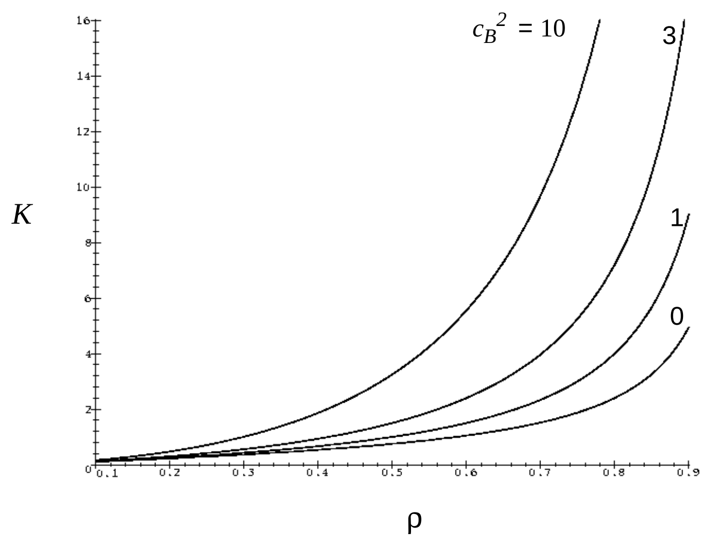
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

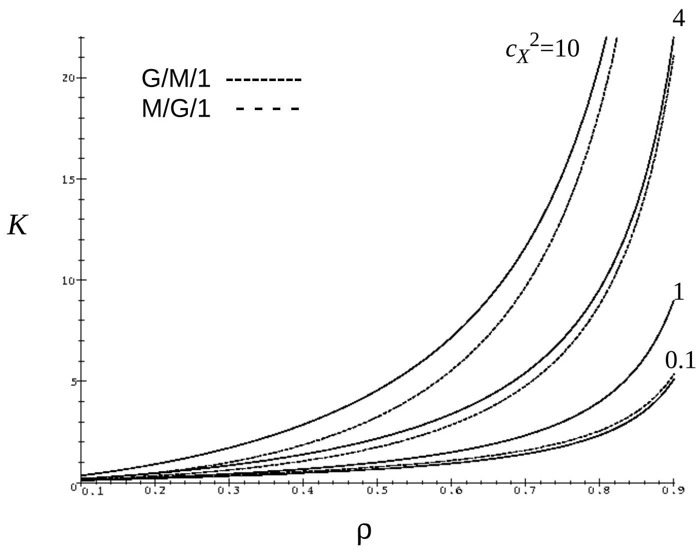
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

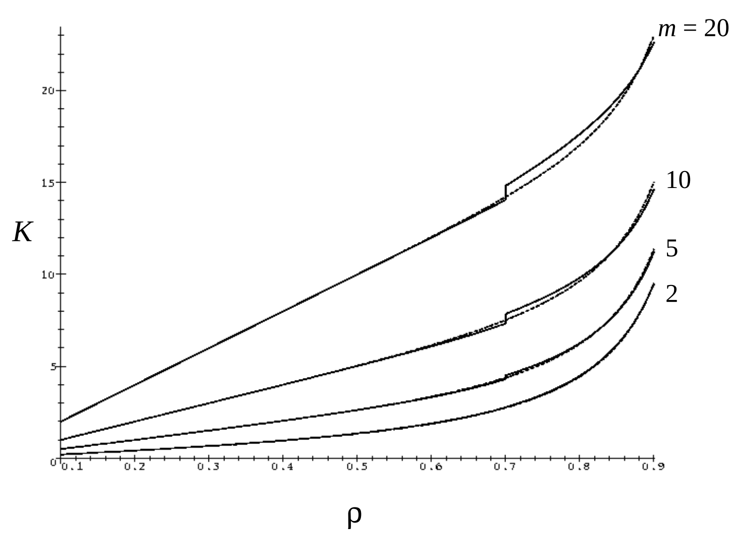
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m+\rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

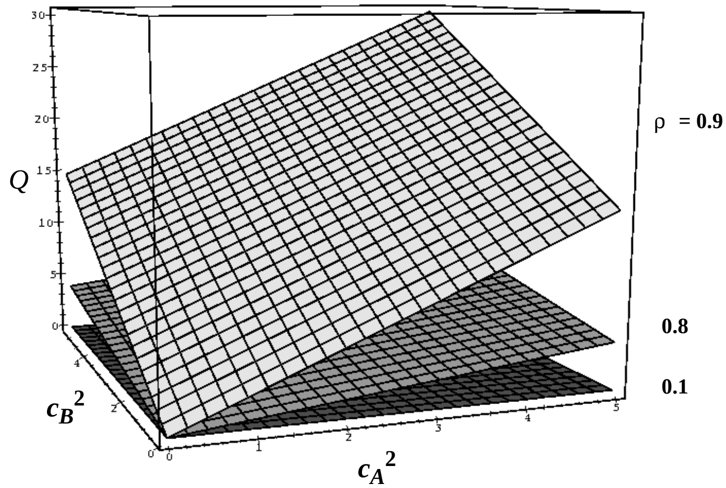
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

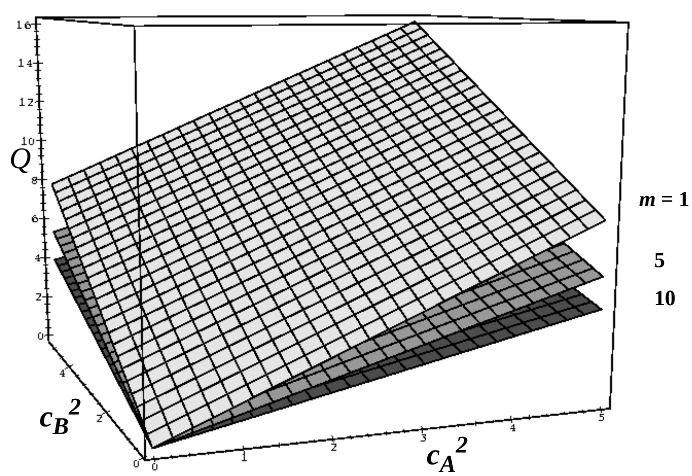
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\text{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

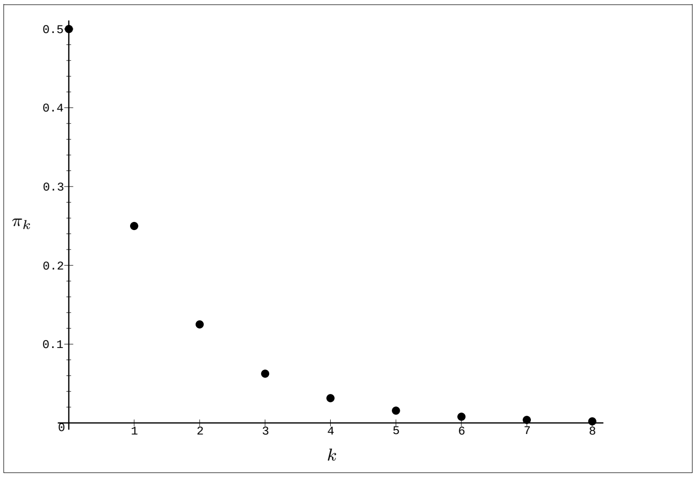


• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

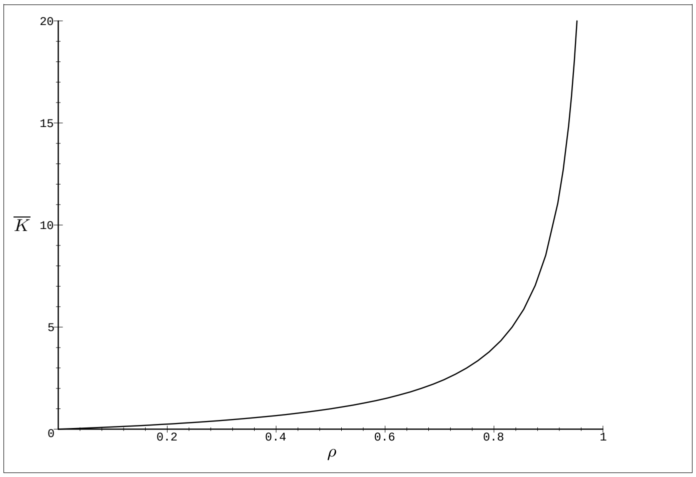
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k = \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

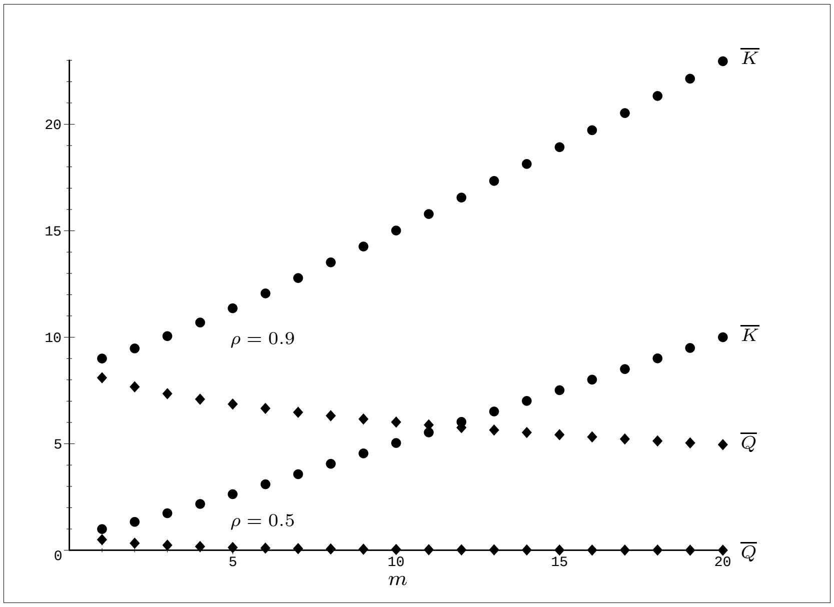
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

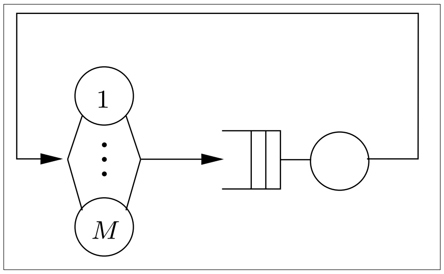
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda / \mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

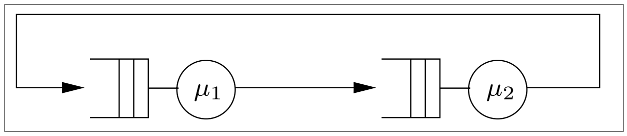
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* 1$)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2}\,,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}}\,, & u \neq 1\,, \\ \dfrac{1}{K+1}\,, & u = 1\,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0)\,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}}\,, \quad u \neq 1\,,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

◆ System is given by:

➤ Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$

➤ Service time arbitrarily distributed:

➤ Mean: $\overline{T}_B = 1/\mu$

➤ Coefficient of variation: c_B

➤ Number of servers: $m = 1$

➤ Queueing discipline: FCFS

◆ The mean waiting time of an arriving job in a M/G/1 system has two components:

➤ The mean remaining service time $\overline{W}_0$ of ther job in service (if any).

➤ The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{\boldsymbol{R}} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

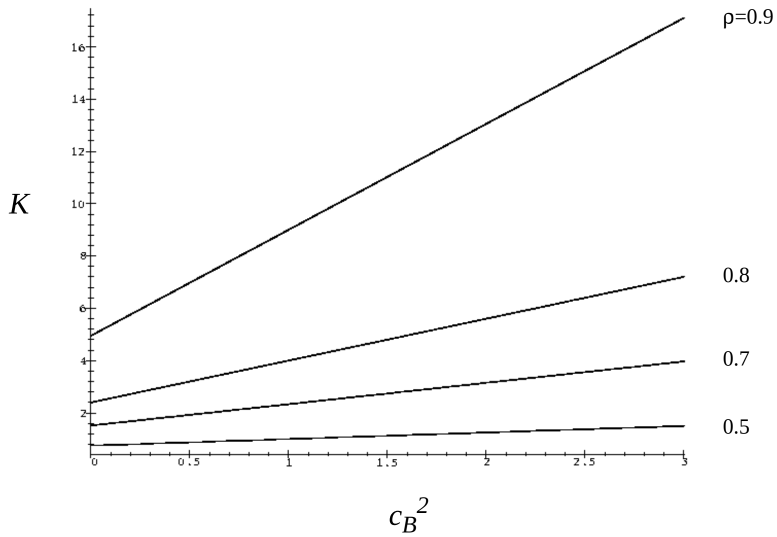
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

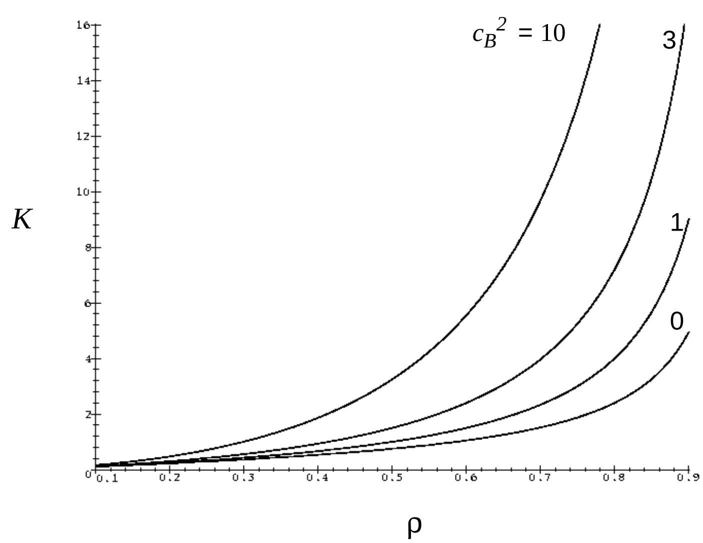
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1 - \sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1 - \sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

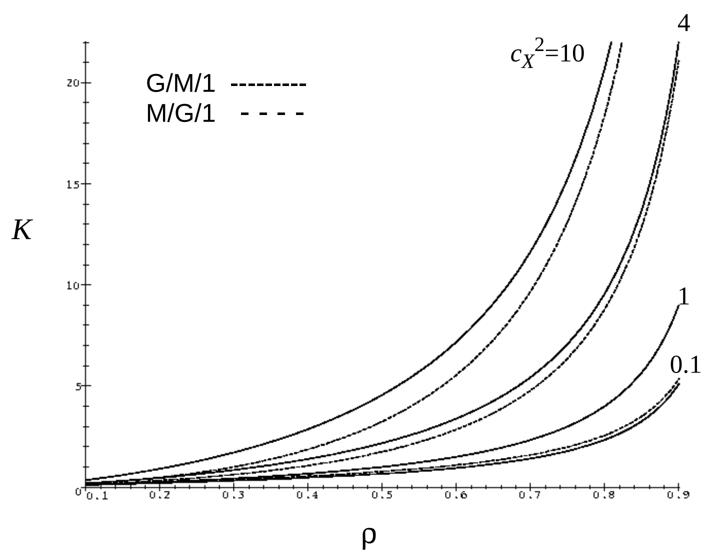
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

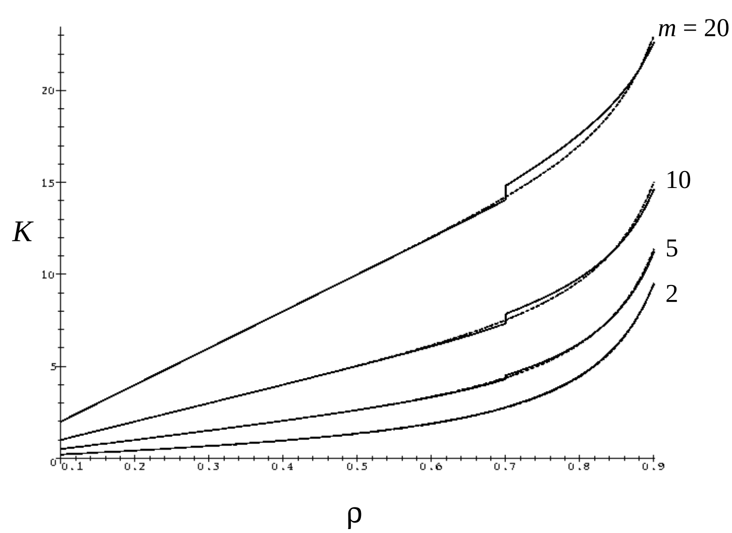
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{\text{ex}}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{\text{app}}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

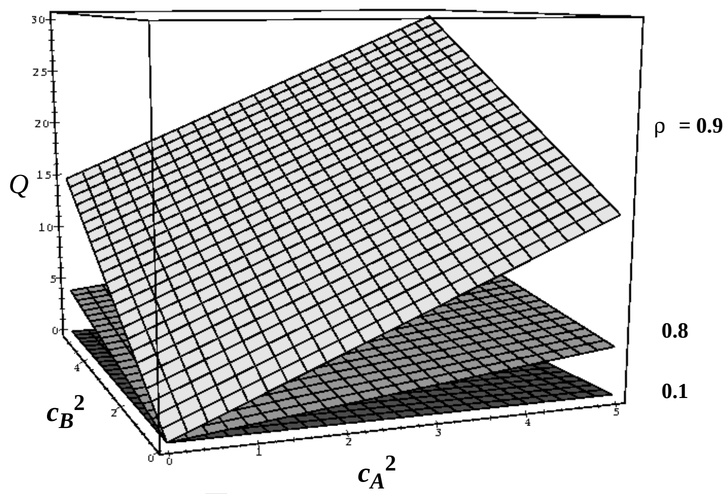
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

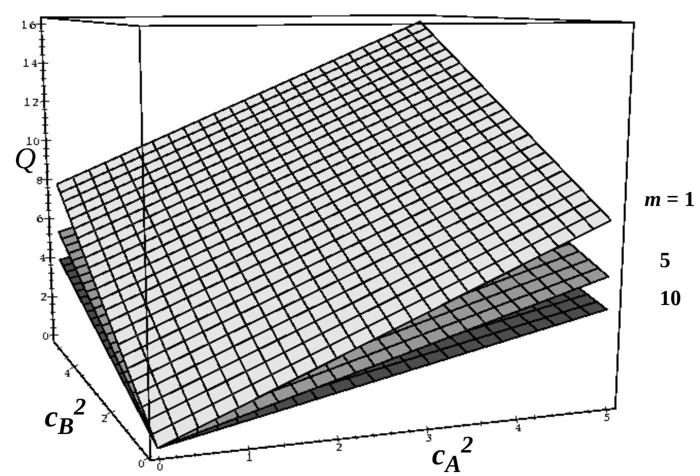
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\dfrac{2}{3}\dfrac{1-\rho}{P_m}\dfrac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\dfrac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

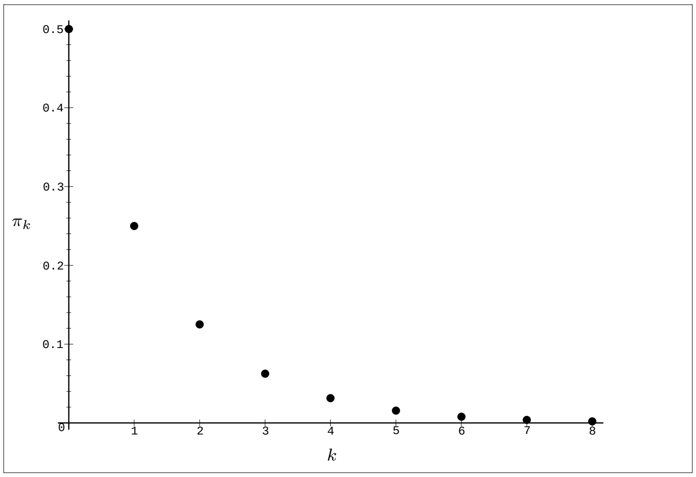


• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

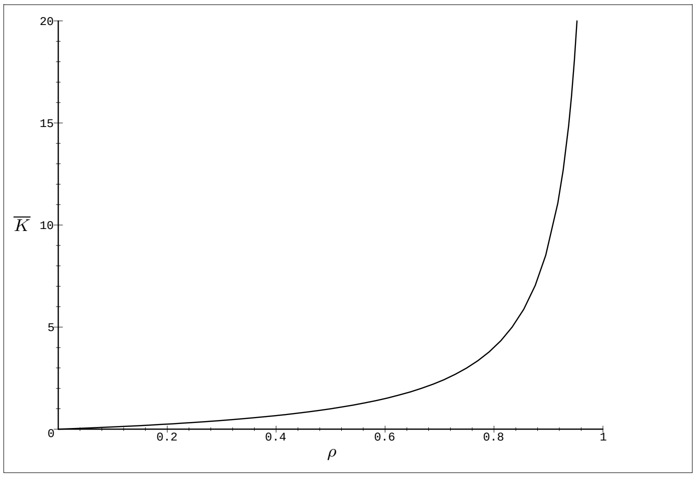
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho} \right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

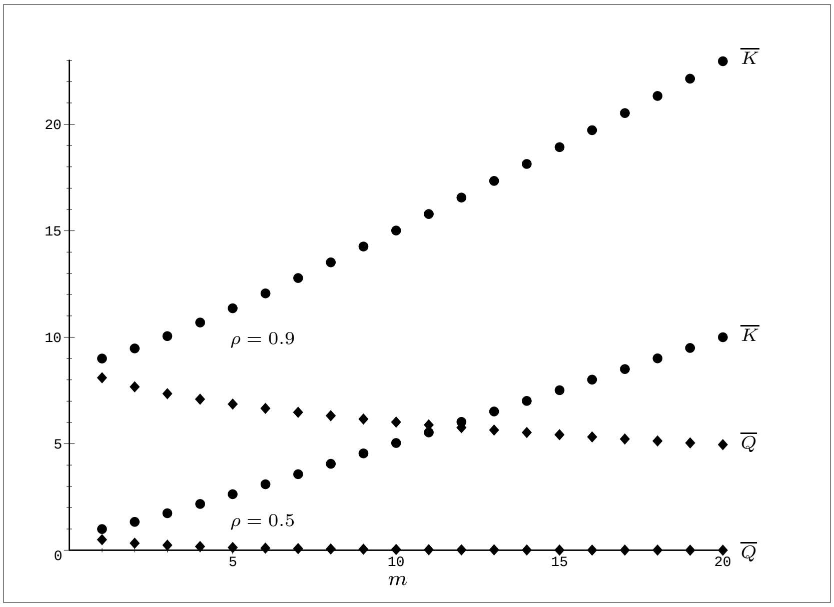
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \frac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

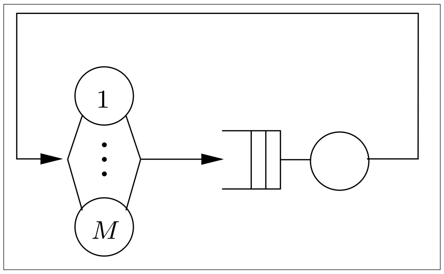
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

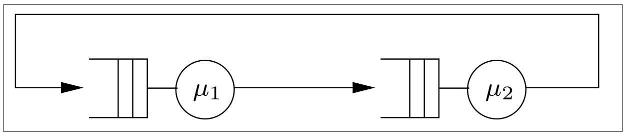
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little’s law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:
  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* 1$)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\[2ex] \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2}\,,$$
$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}}\,, & u \neq 1\,, \\[2ex] \dfrac{1}{K+1}\,, & u = 1\,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0)\,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}}\,, \quad u \neq 1\,,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

  And with Little's law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

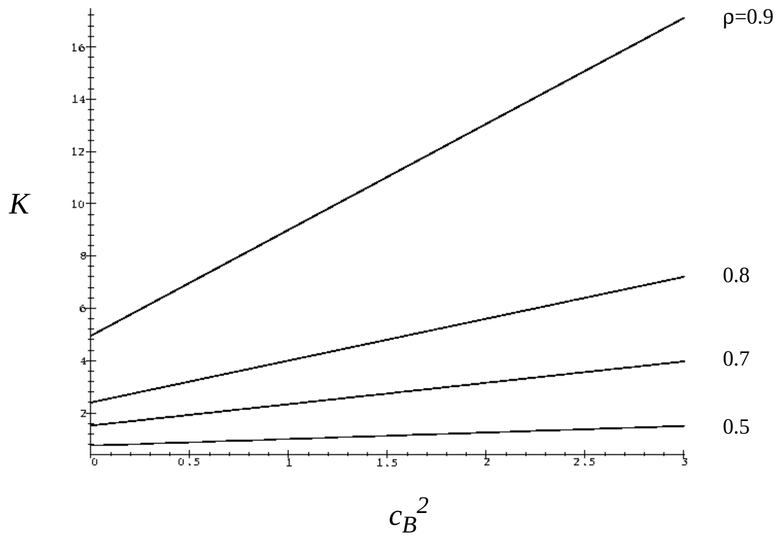
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

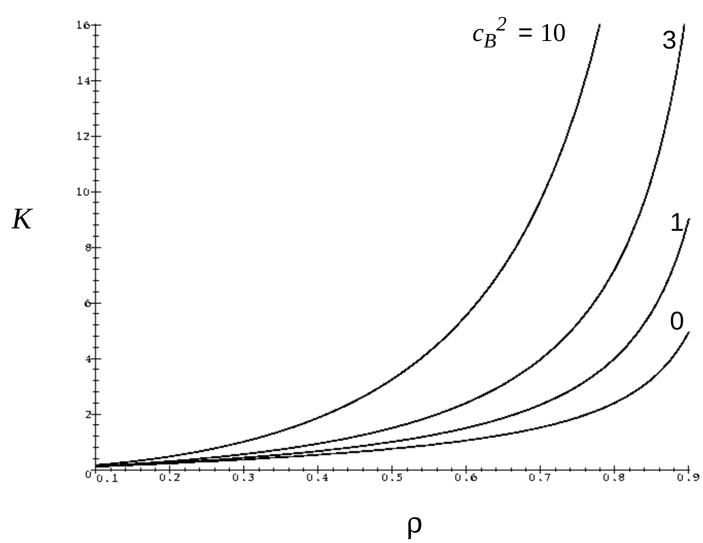
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)} .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

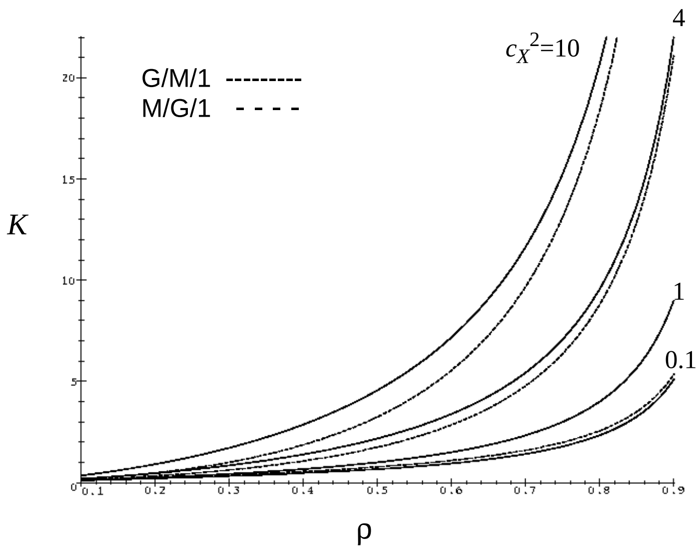
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\text{KLB}}$$

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

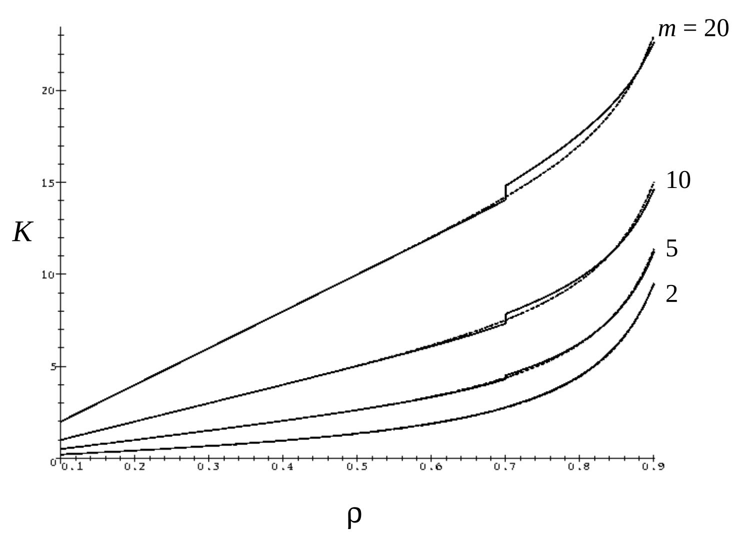
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

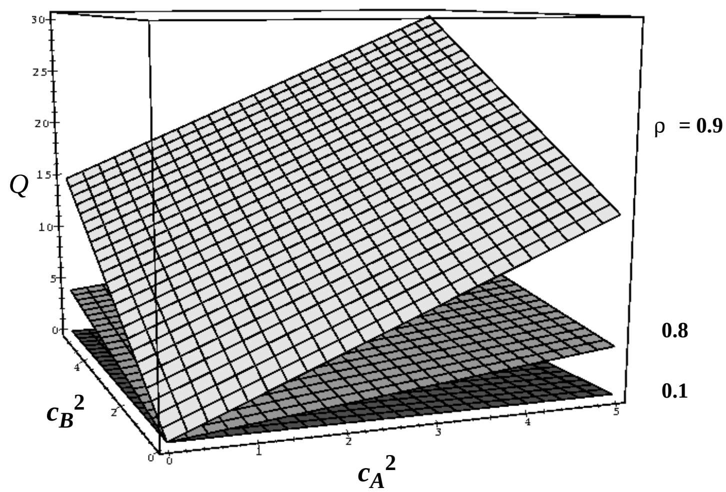
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

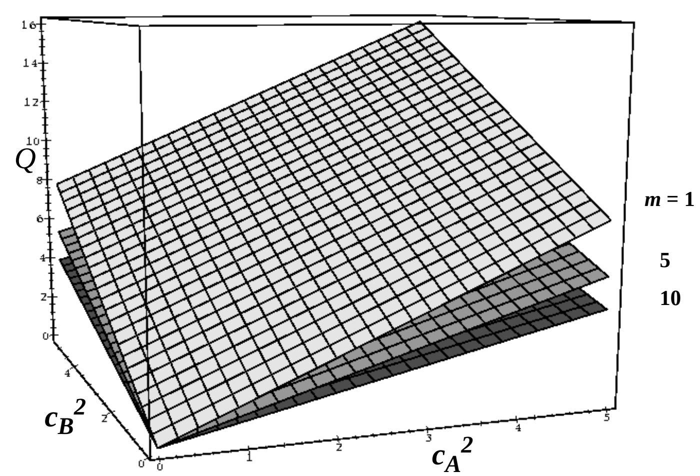
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



- Mean queue length $\overline{Q}$ of a G/G/10- system.

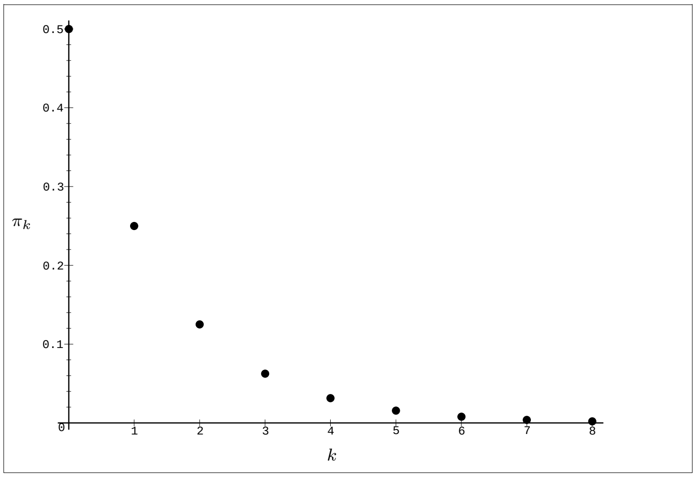


- Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

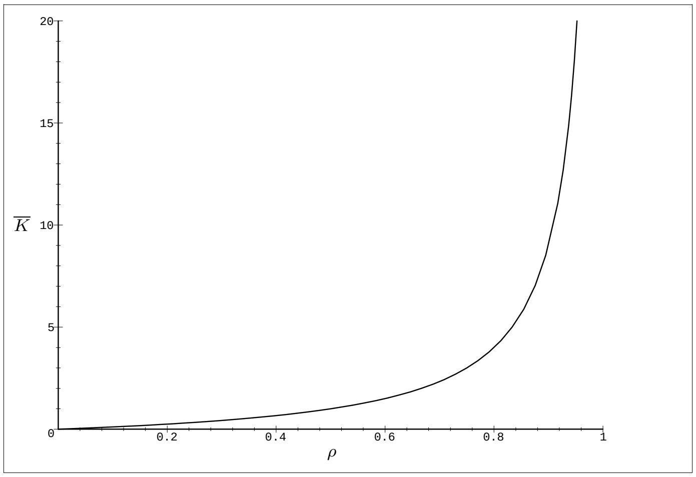
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k = \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

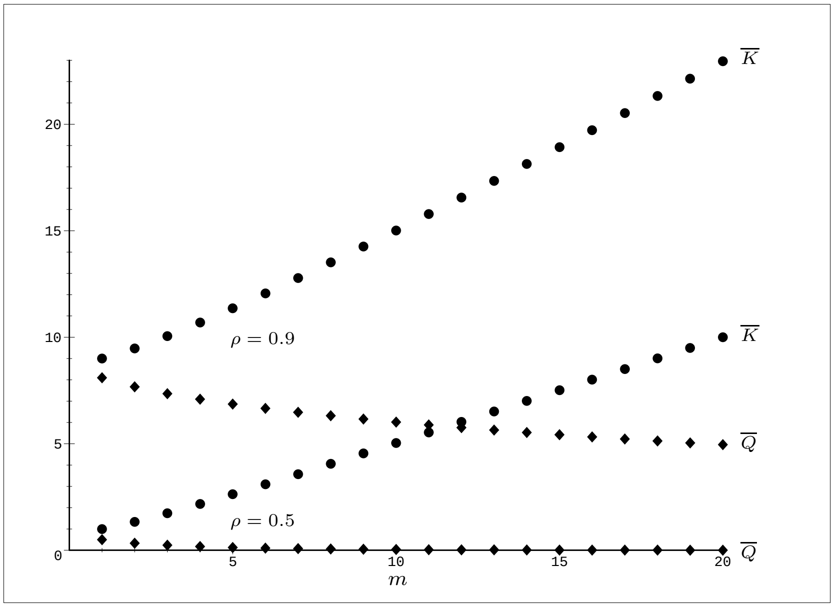
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \frac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

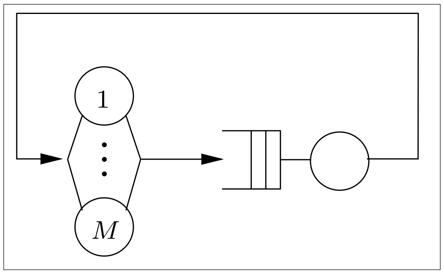
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum\limits_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

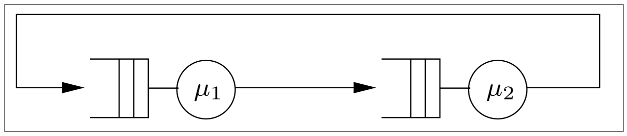
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little’s law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:
  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1-\frac{1}{u}}{1-\left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\[2ex] \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1/\mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} \,,$$
$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} \,, & u \neq 1 \,, \\ \dfrac{1}{K+1} \,, & u = 1 \,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) \,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} \,, \quad u \neq 1 \,,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

  And with Little's law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

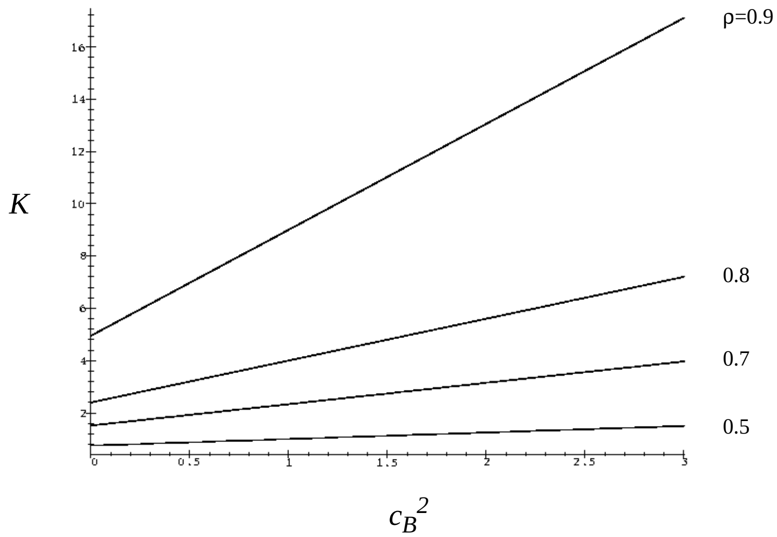
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

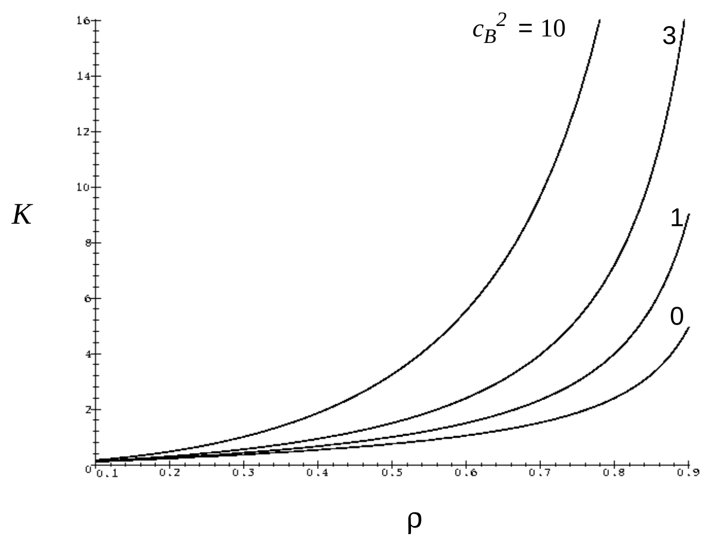
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)} \, .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

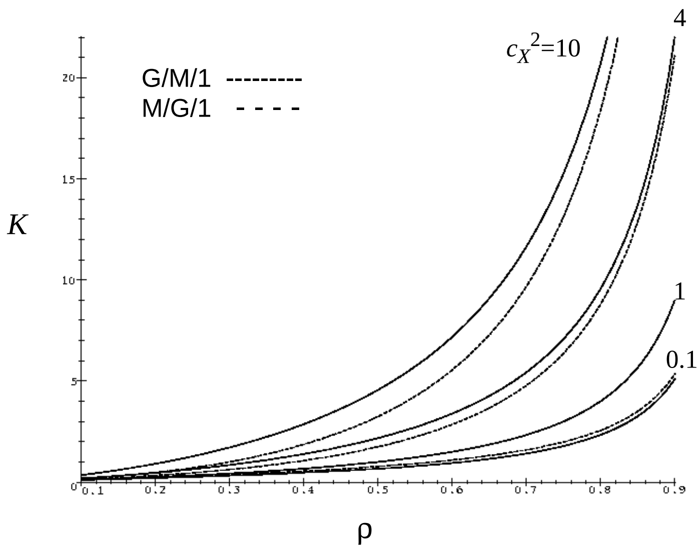
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\text{KLB}}$$

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

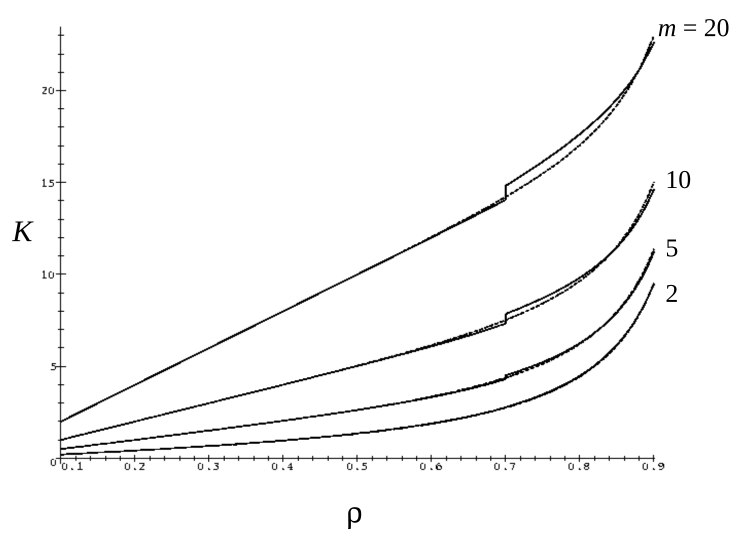
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

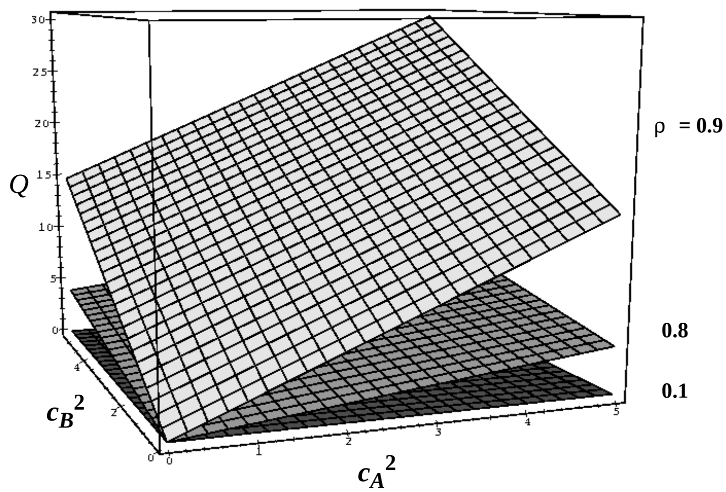
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

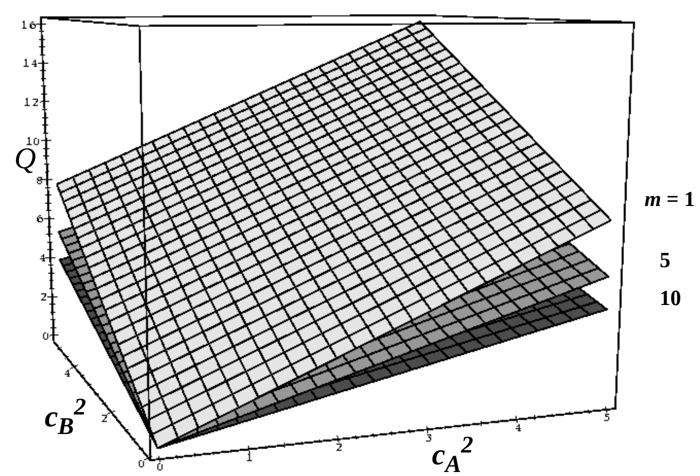
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

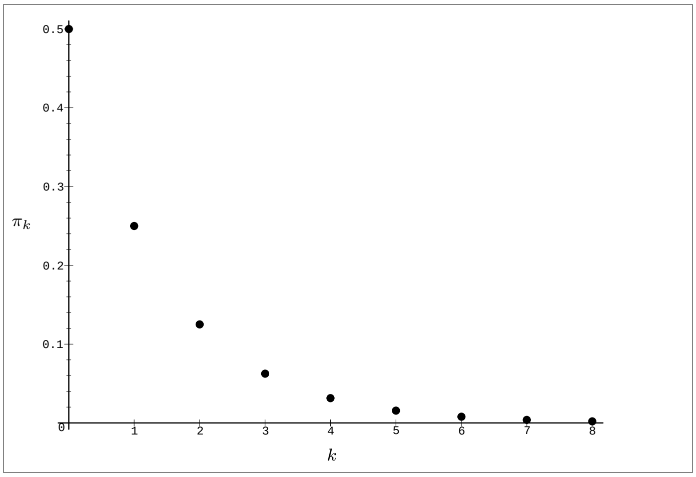


• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

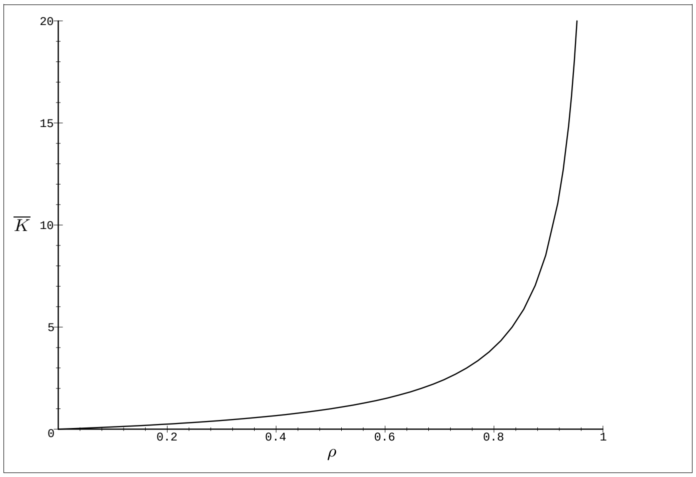
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k = \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

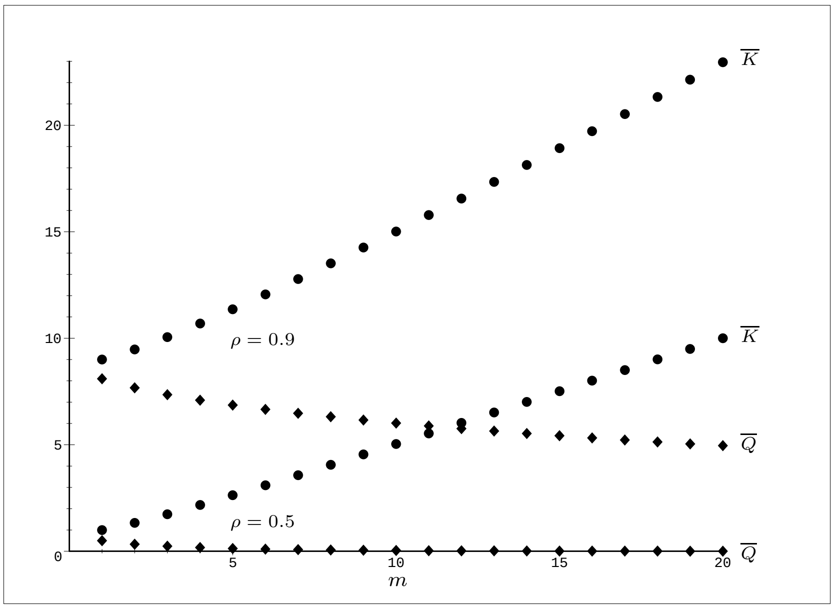
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K\,, \\ 0\,, & k > K\,. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k\,, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

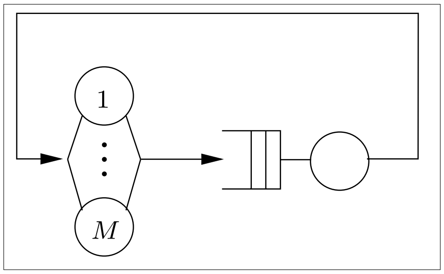
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

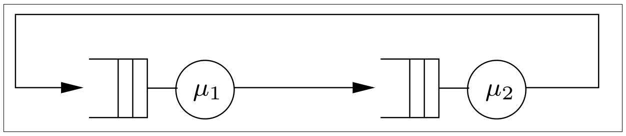
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2}\,,$$
$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}}\,, & u \neq 1\,, \\[2ex] \dfrac{1}{K+1}\,, & u = 1\,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0)\,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}}\,, \quad u \neq 1\,,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

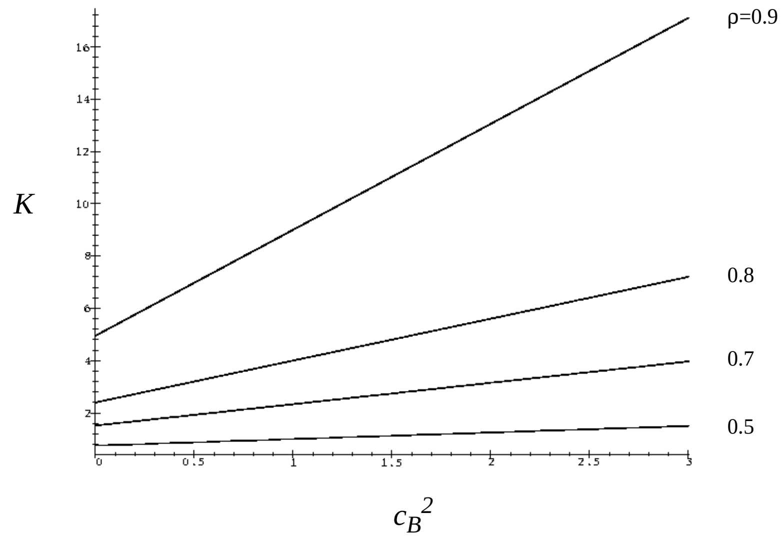
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

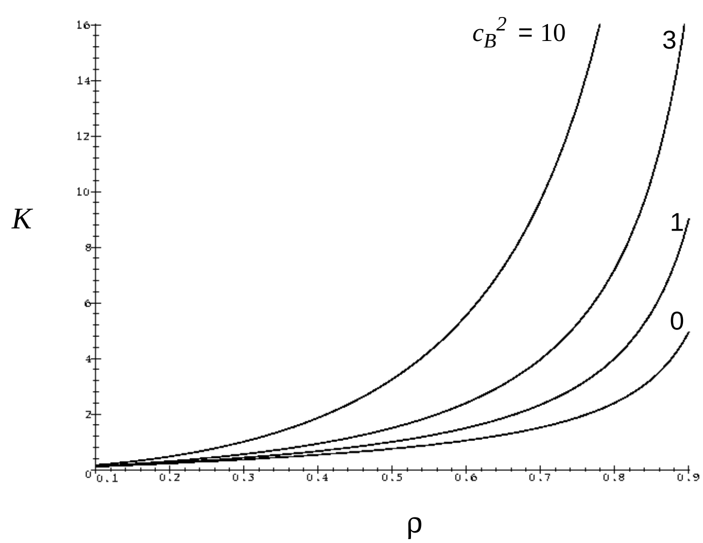
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)} .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

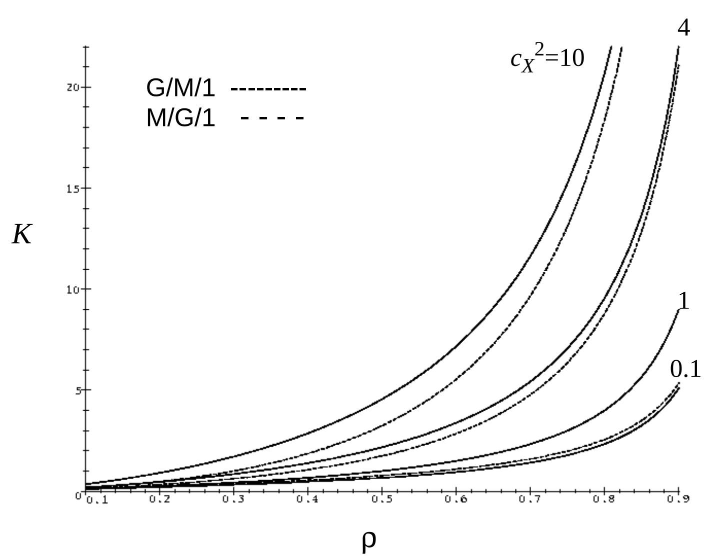
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

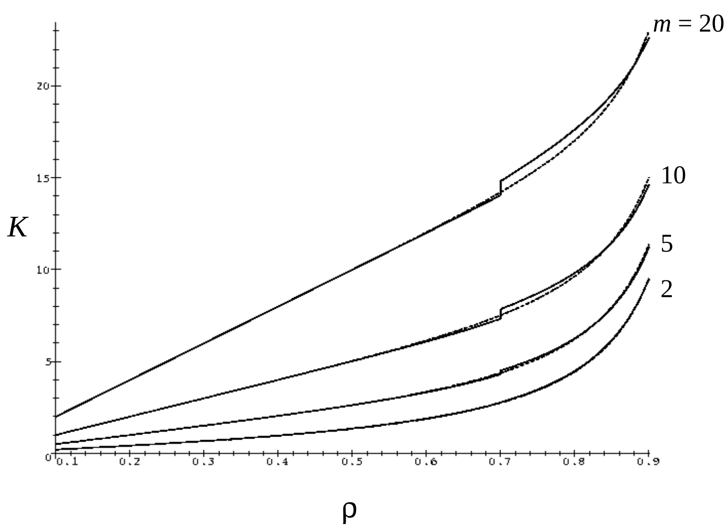
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7\,, \\[2ex] \rho^{\frac{m+1}{2}}, & \rho < 0.7\,. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{\mathrm{ex}}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{\mathrm{app}}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

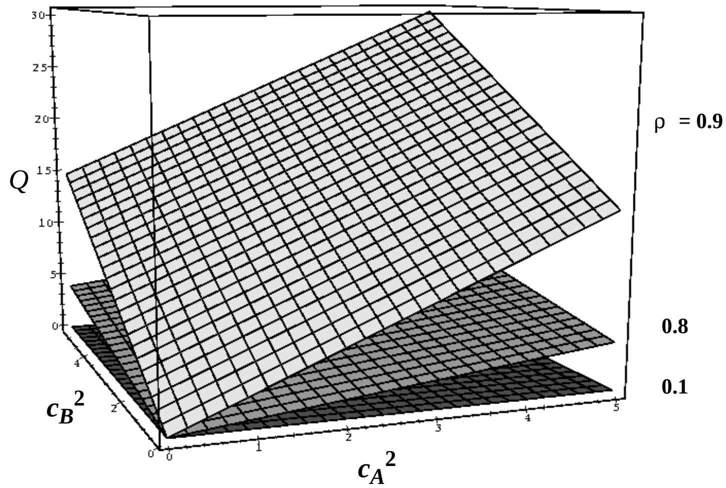
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

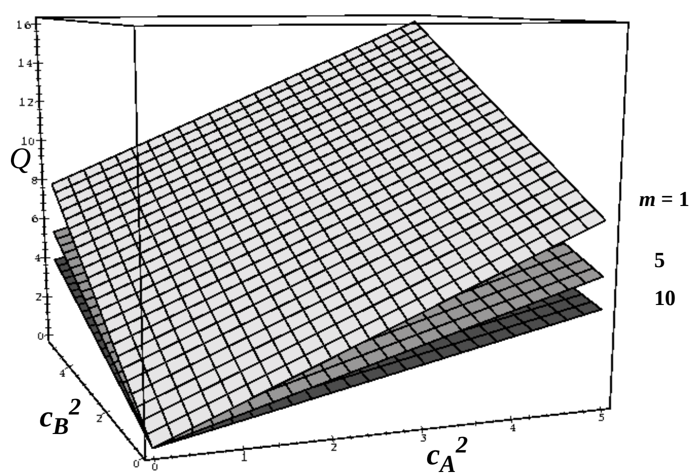
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

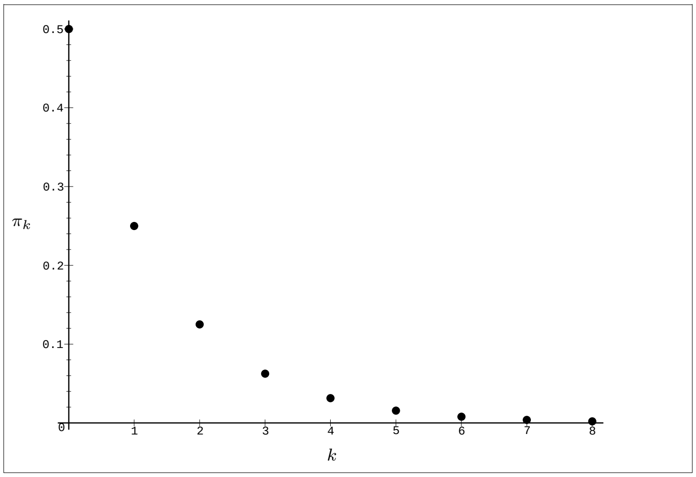


• Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

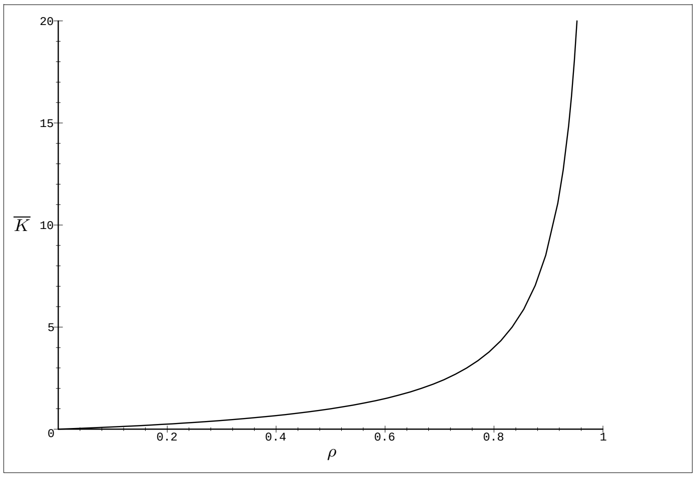
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}\frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

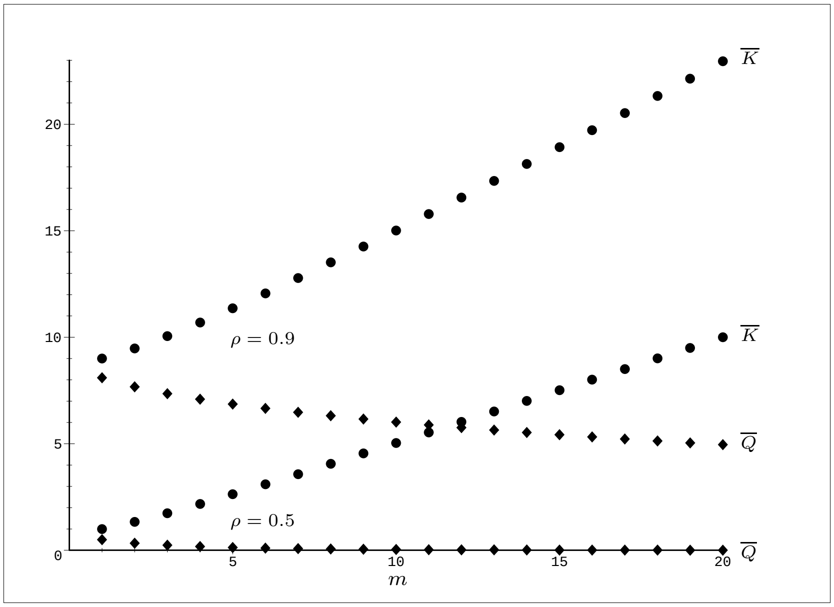
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m \,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x} \,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

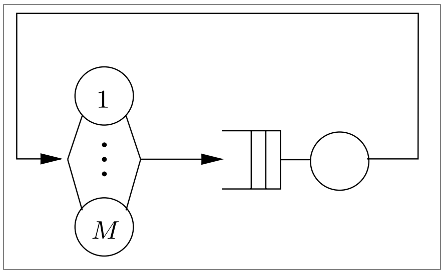
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum\limits_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

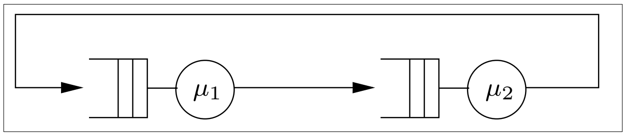
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:
  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten* 2)):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} \,,$$
$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} \,, & u \neq 1 \,, \\[2ex] \dfrac{1}{K+1} \,, & u = 1 \,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) \,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1 \mu_1 = \rho_2 \mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} \,, \quad u \neq 1 \,,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{\boldsymbol{R}} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\text{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

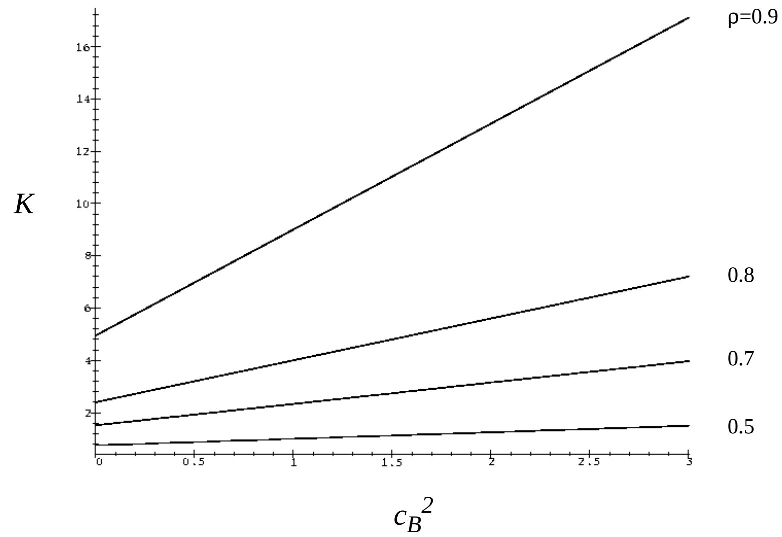
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

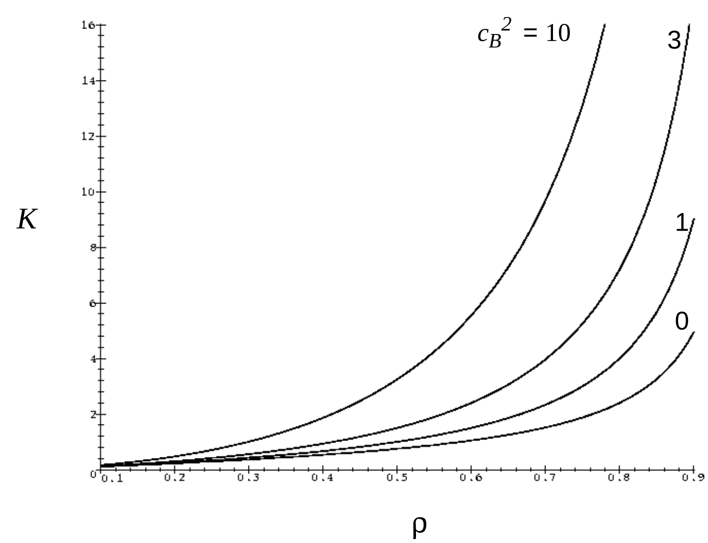
$$\overline{Q}_{\text{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\text{M/D/1}} = \frac{\rho^2}{2(1-\rho)} \,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: m = 1

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

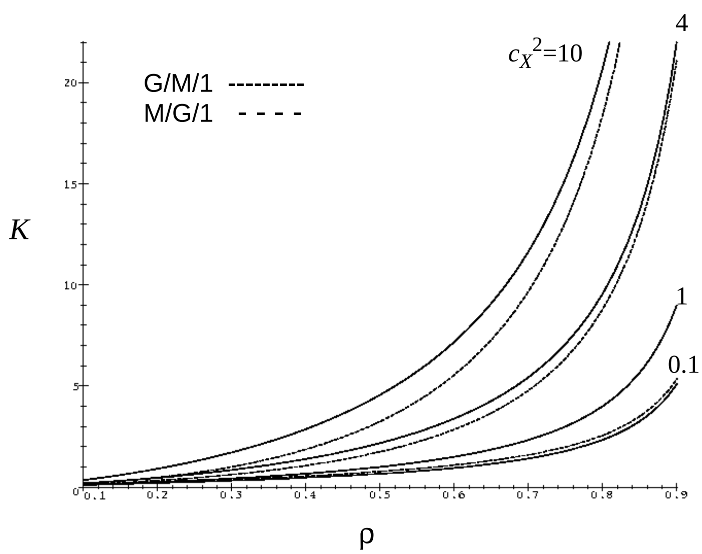
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\text{KLB}}$$

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

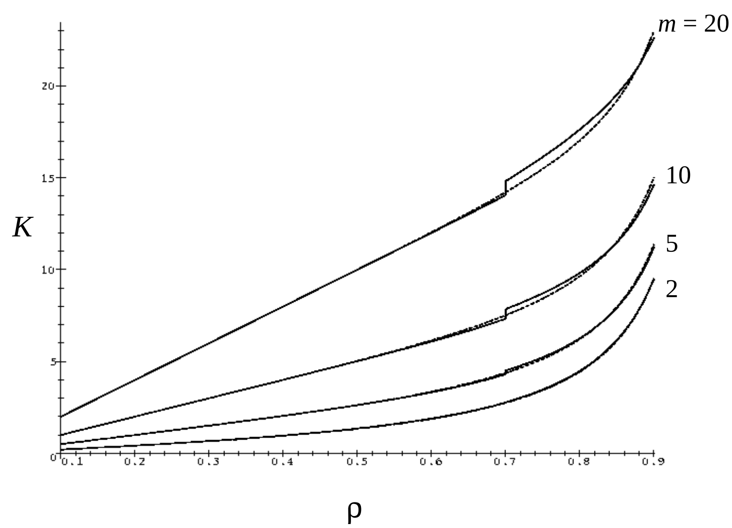
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

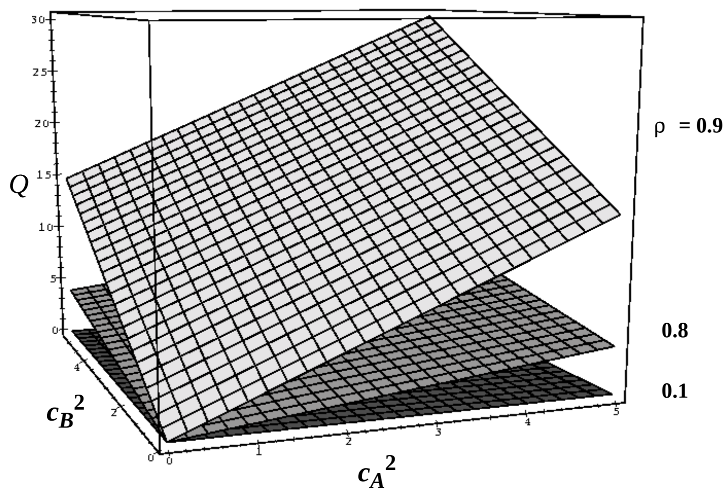
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

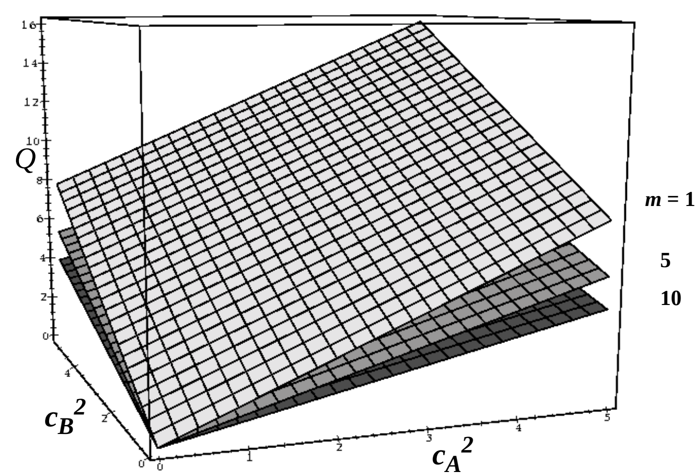
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

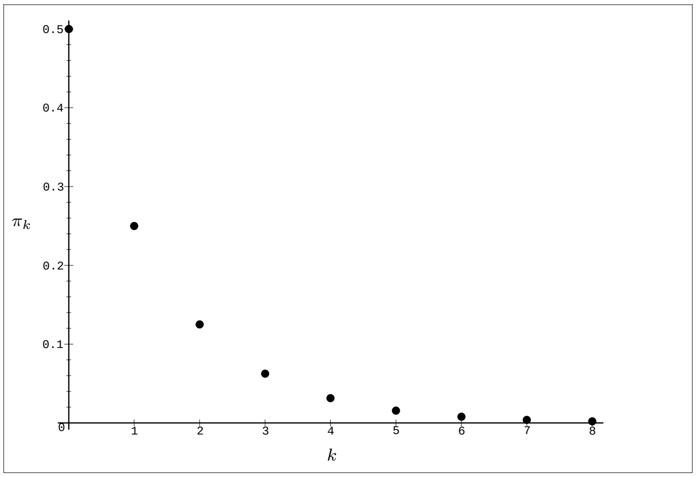


• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

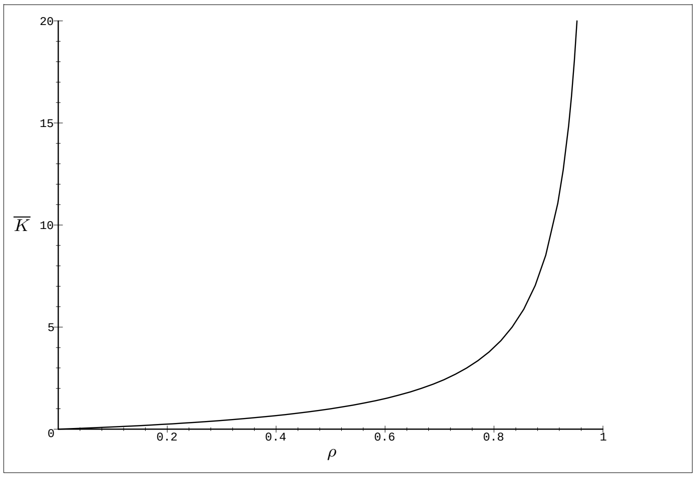
1 M/M/1

Steady-State Probability:
$$\pi_k = (1-\rho)\rho^k$$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}\frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

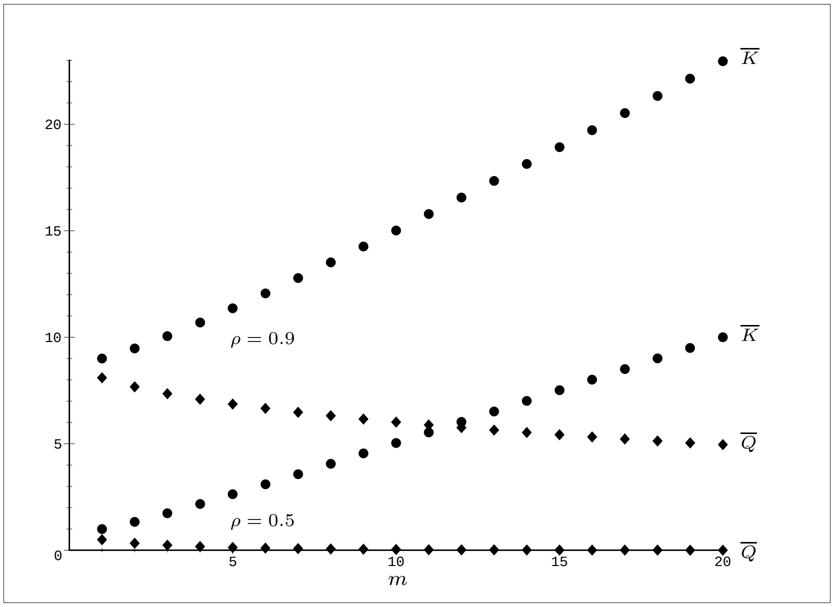
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m \,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x} \,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

➤ Infinite Server means, that the number of servers is at least the number of jobs in the system

➤ No job has to wait

➤ Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

➤ Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

➤ Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

➤ Queue has a finite capacity

➤ The maximum number of jobs in the system is K

➤ An arriving job is lost, if the queue is full

➤ Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \frac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

➤ For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

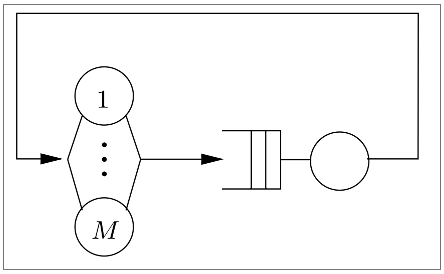
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum\limits_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

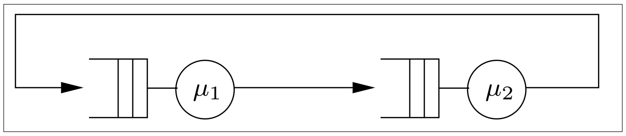
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:
  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} \,,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} \,, & u \neq 1 \,, \\ \dfrac{1}{K+1} \,, & u = 1 \,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) \,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}} \,, \quad u \neq 1 \,,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda / \mu$

$$\overline{W} = \frac{\overline{W}_0}{1 - \rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{R} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{R}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

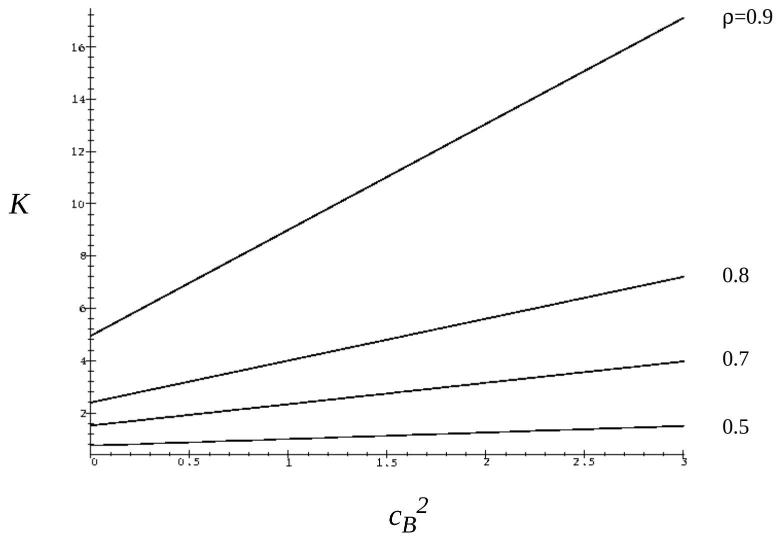
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

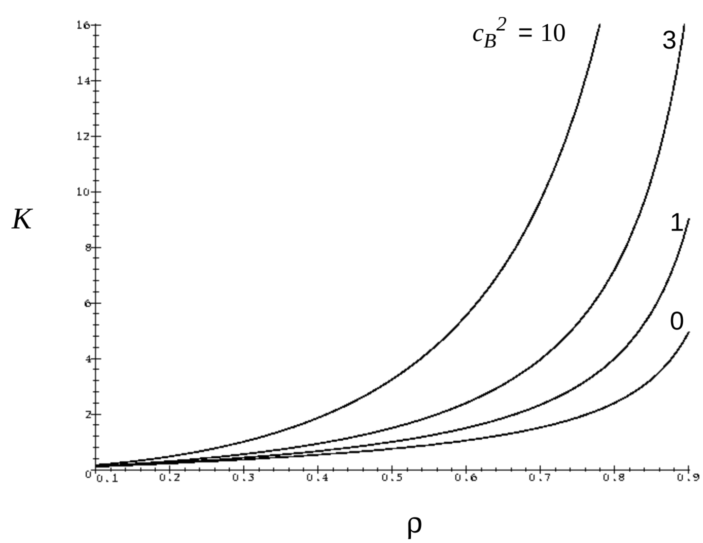
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\ .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1 - \sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1 - \sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda / (s + \lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

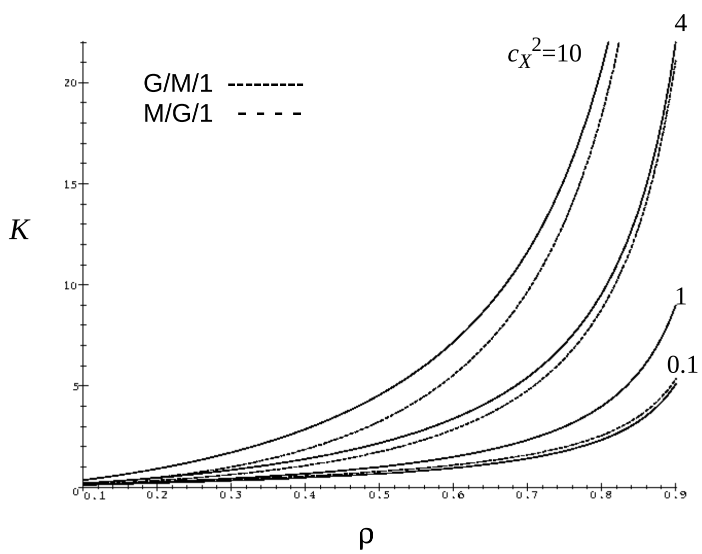
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s + \lambda} \right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\text{KLB}}$$

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

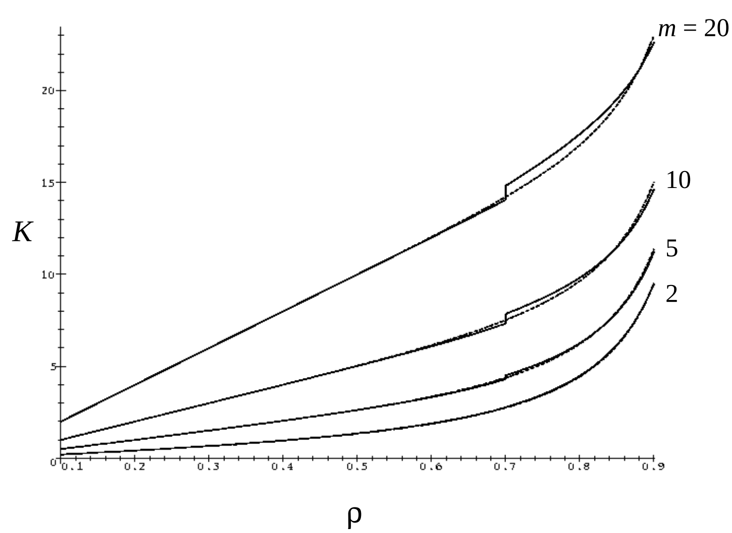
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7\,, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7\,. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

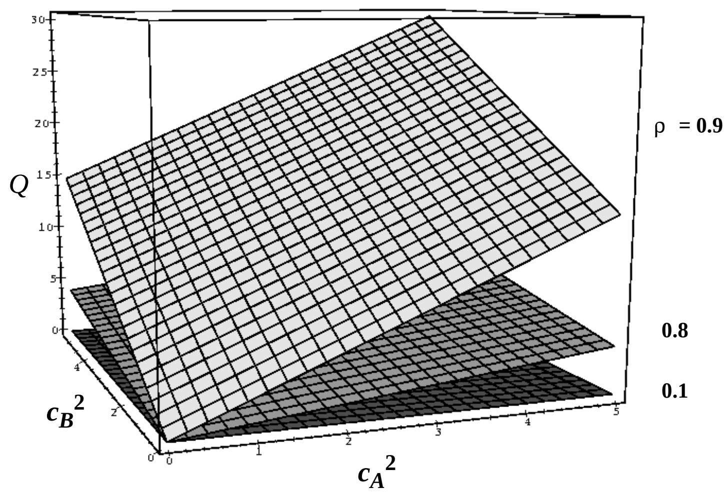
➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

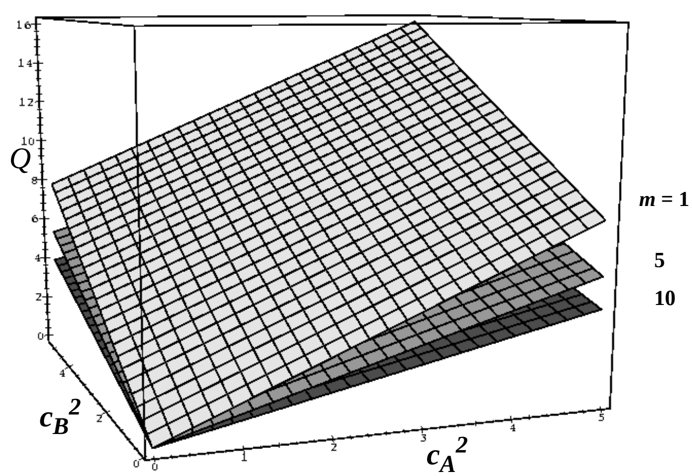
D.4 FIFO-Systems

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



- Mean queue length $\overline{Q}$ of a G/G/10- system.

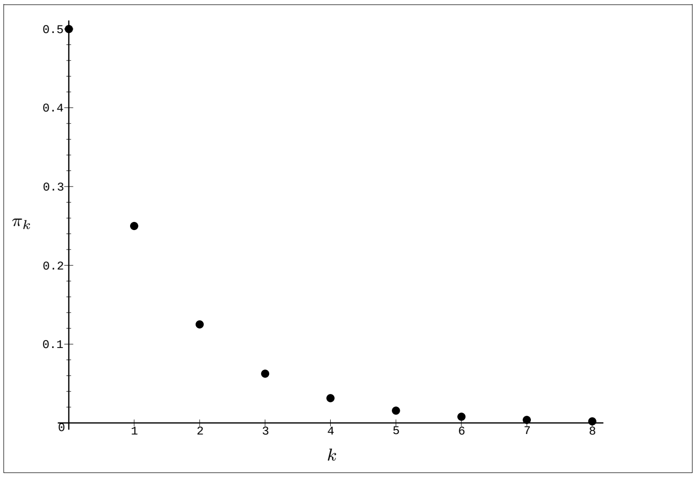


- Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

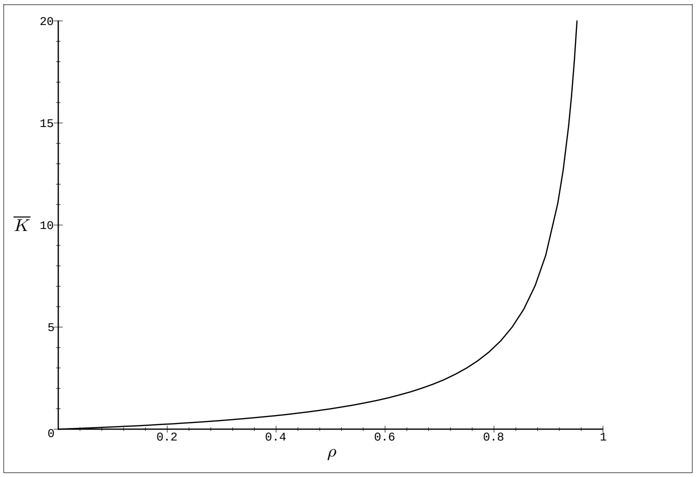
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

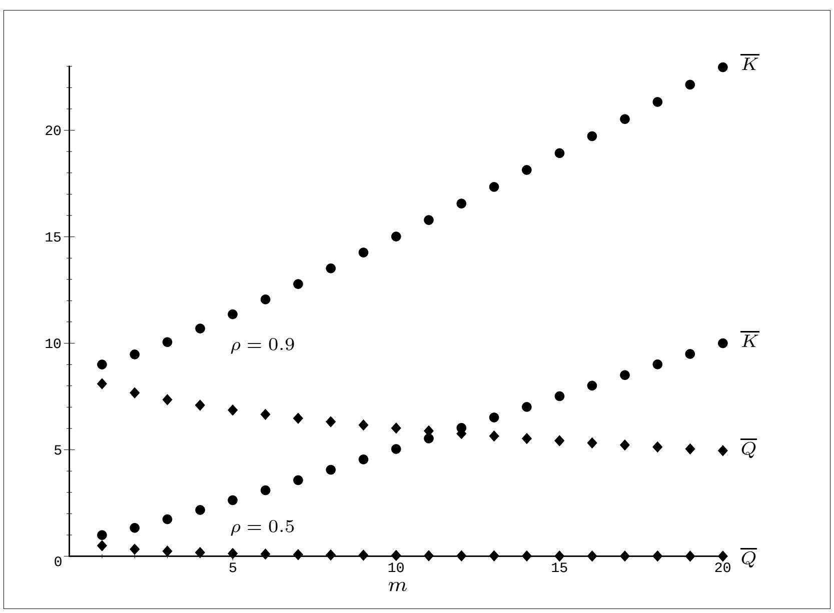
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m \,, & x = 0 \,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x} \,, & x > 0 \,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

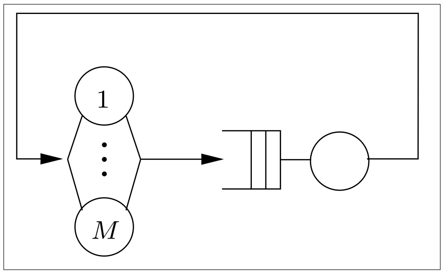
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

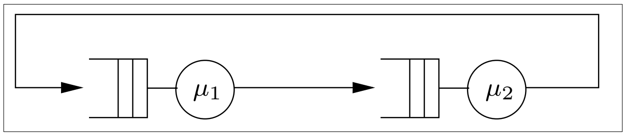
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* 1$)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} ,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} , & u \neq 1 , \\ \dfrac{1}{K+1} , & u = 1 . \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) , \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}} , \quad u \neq 1 ,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

◆ System is given by:

➤ Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$

➤ Service time arbitrarily distributed:

➤ Mean: $\overline{T}_B = 1/\mu$

➤ Coefficient of variation: c_B

➤ Number of servers: $m = 1$

➤ Queueing discipline: FCFS

◆ The mean waiting time of an arriving job in a M/G/1 system has two components:

➤ The mean remaining service time $\overline{W}_0$ of ther job in service (if any).

➤ The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little’s law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

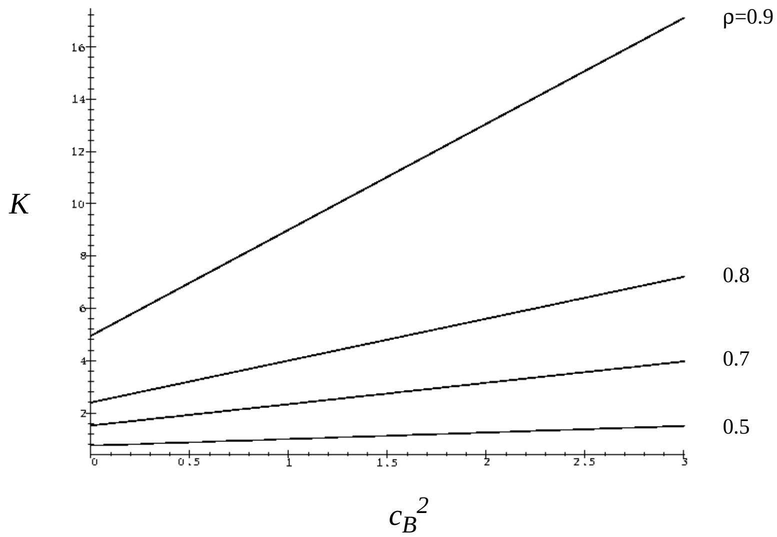
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

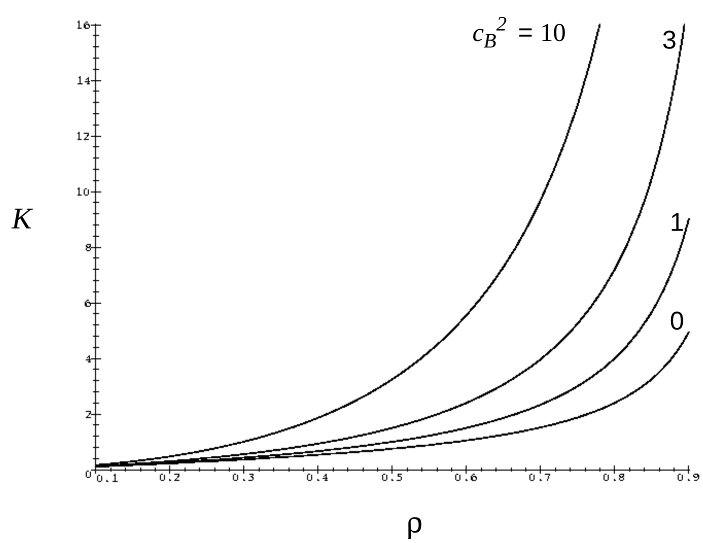
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)} .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

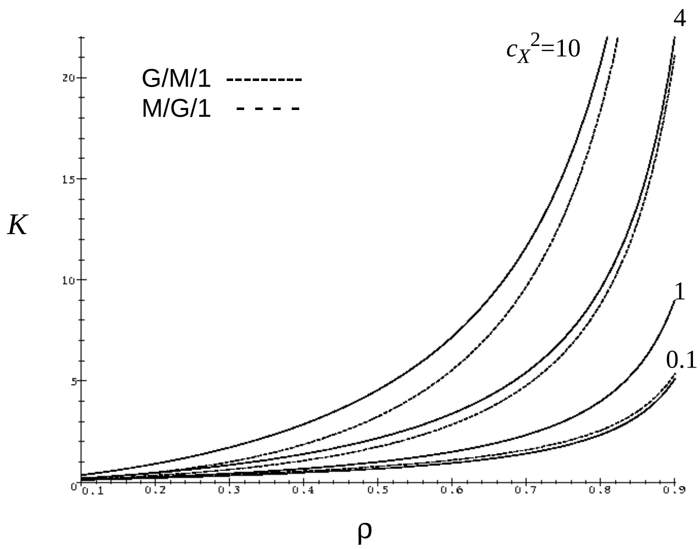
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

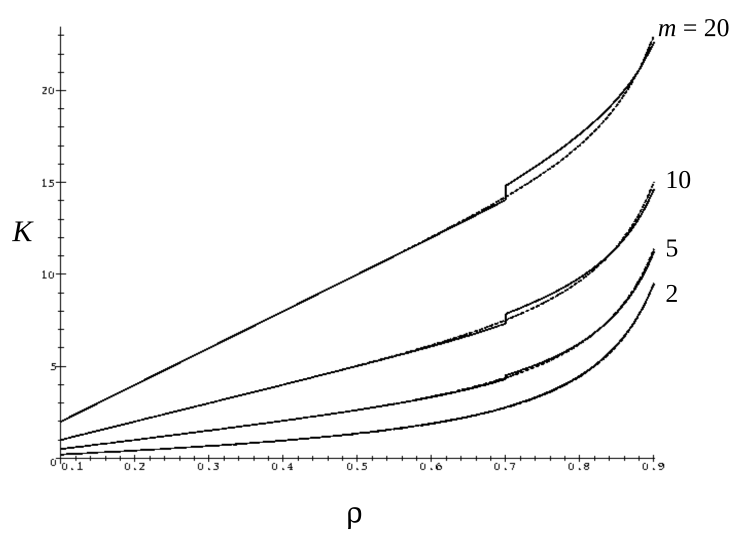
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m+\rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

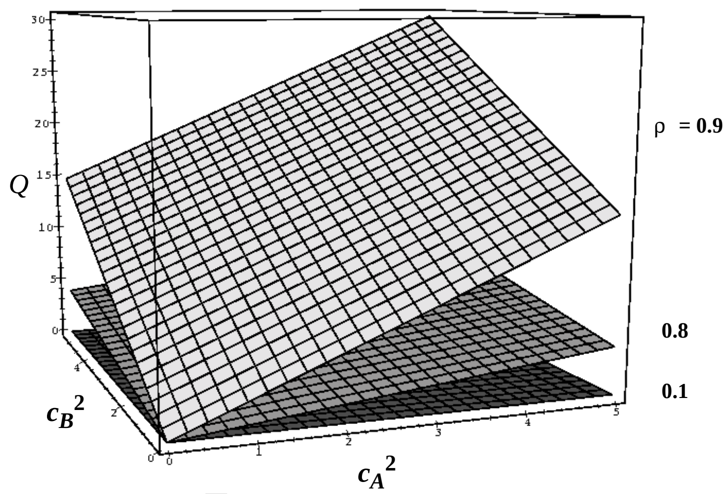
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

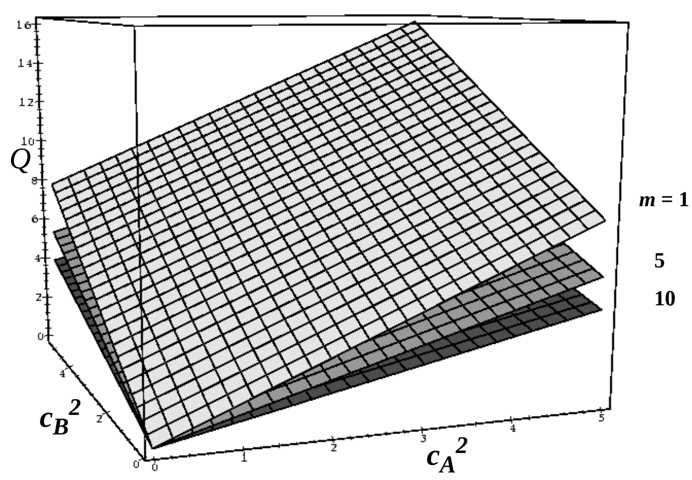
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

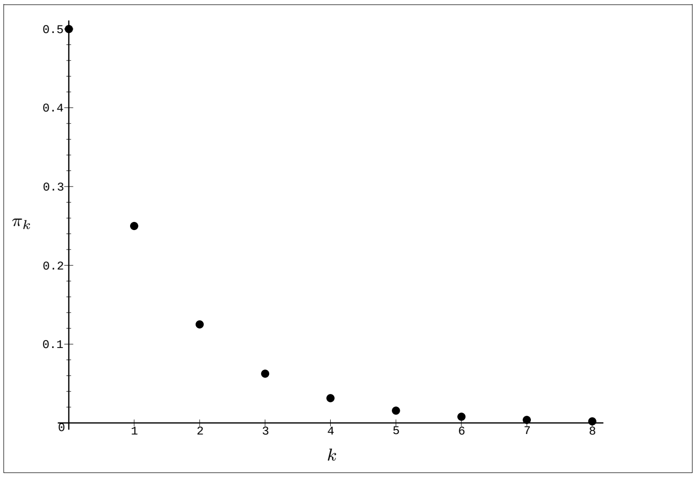


• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

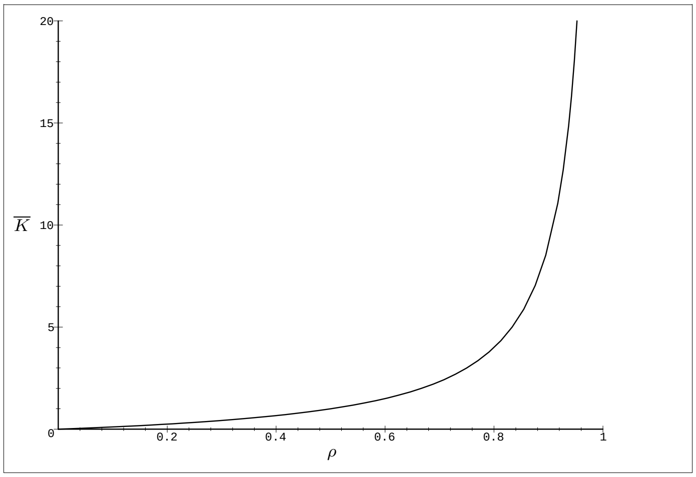
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\operatorname{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k = \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

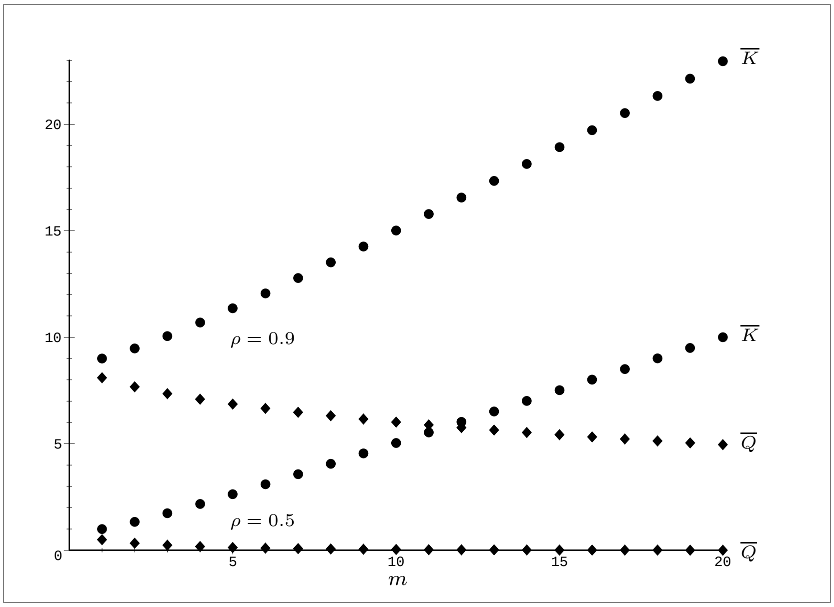
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot e^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

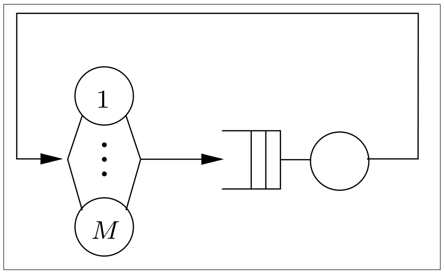
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

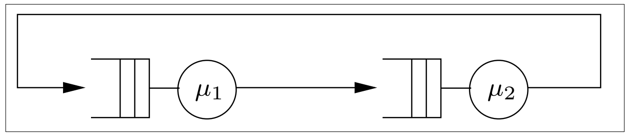
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2}\,,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}}\,, & u \neq 1\,, \\[2ex] \dfrac{1}{K+1}\,, & u = 1\,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0)\,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}}\,, \quad u \neq 1\,,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

◆ System is given by:

➤ Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$

➤ Service time arbitrarily distributed:

➤ Mean: $\overline{T}_B = 1/\mu$

➤ Coefficient of variation: c_B

➤ Number of servers: $m = 1$

➤ Queueing discipline: FCFS

◆ The mean waiting time of an arriving job in a M/G/1 system has two components:

➤ The mean remaining service time $\overline{W}_0$ of ther job in service (if any).

➤ The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little's law: $\overline{Q} = \lambda \overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\, \overline{\boldsymbol{R}} + P(\text{server is idle})\, 0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System ($c_B = 1$) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

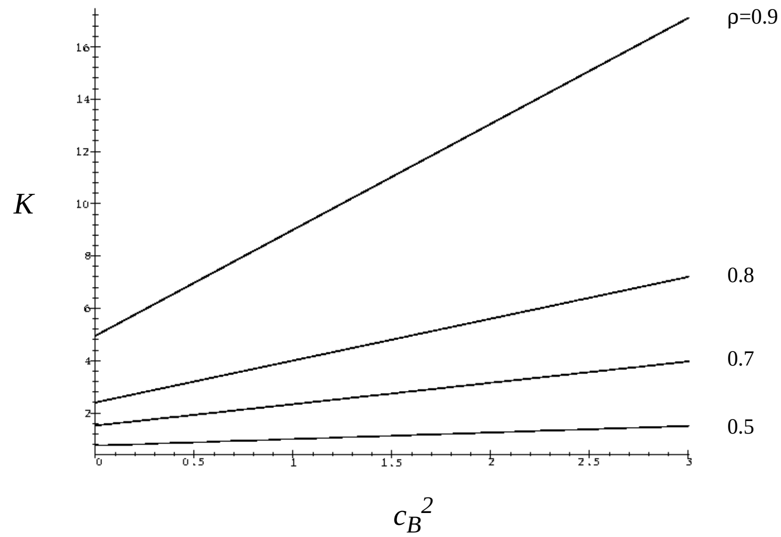
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems ($c_B = 1$) and M/D/1-systems ($c_B = 0$) we obtain:

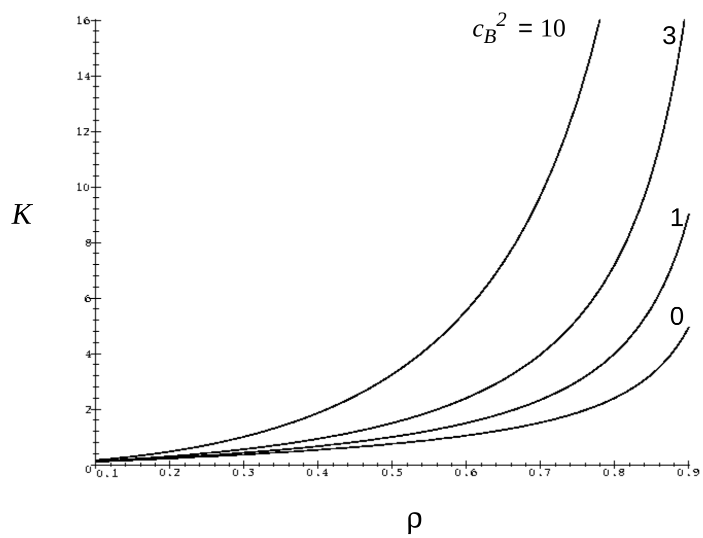
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)} .$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

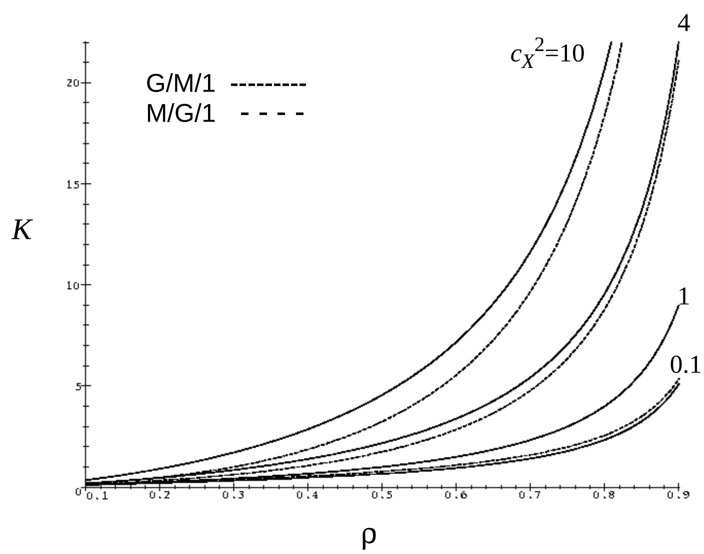
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

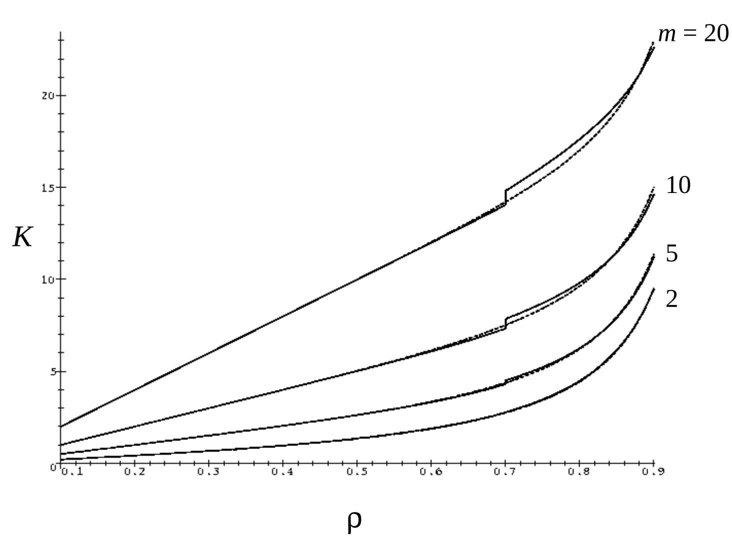
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m+\rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

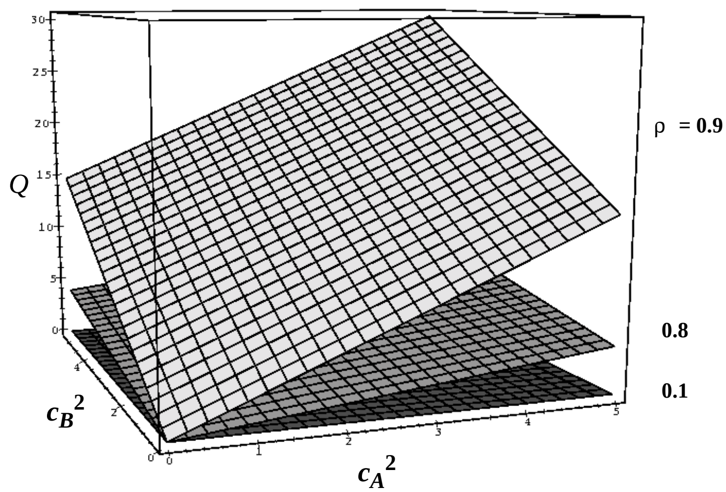
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

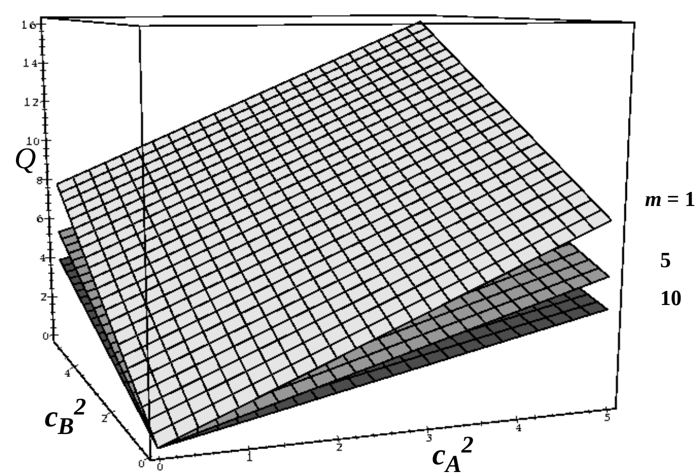
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.

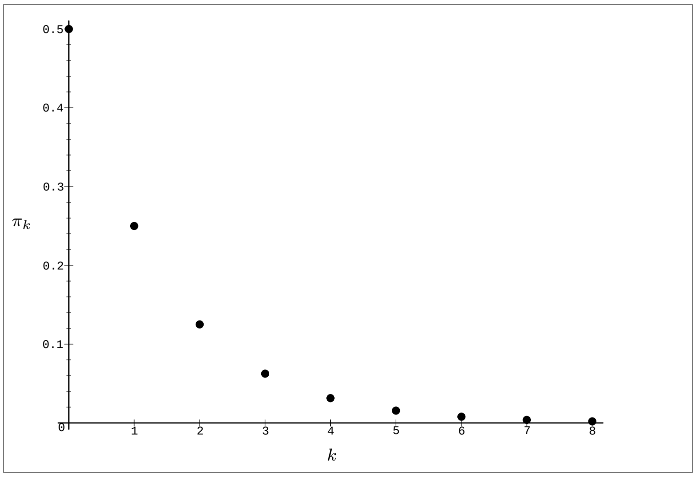


• Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

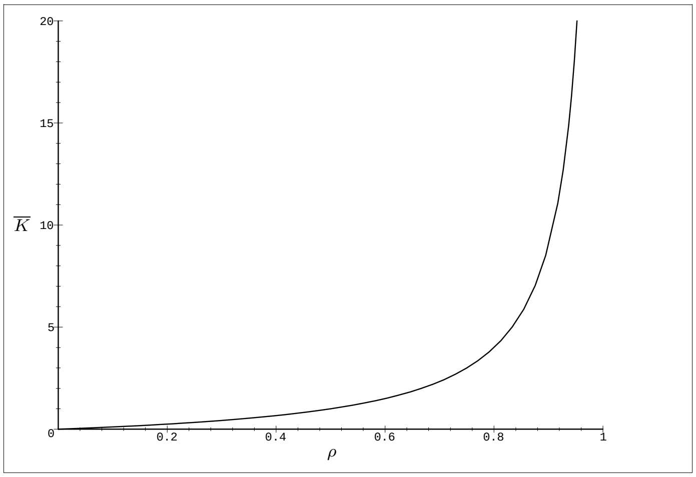
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



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➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\[2ex] \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho} \right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

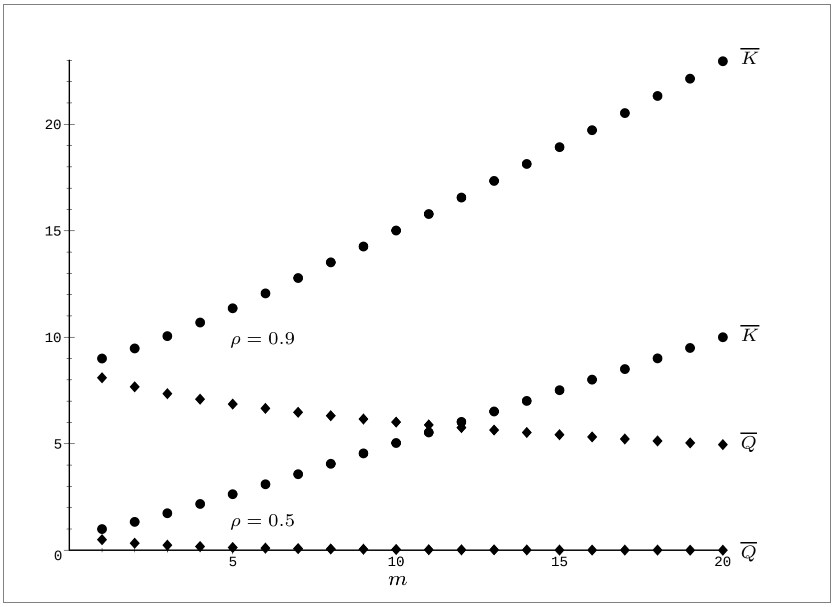
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

➤ Infinite Server means, that the number of servers is at least the number of jobs in the system

➤ No job has to wait

➤ Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

➤ Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

➤ Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

➤ Queue has a finite capacity

➤ The maximum number of jobs in the system is K

➤ An arriving job is lost, if the queue is full

➤ Steady-state probability (a = λ/μ):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

➤ For λ = μ (a = 1):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

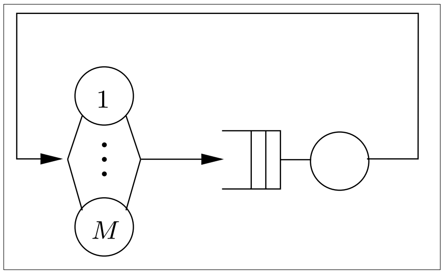
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

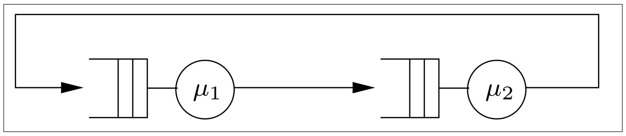
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P(k_1$ *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\ \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten* 2*)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} ,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} , & u \neq 1 , \\ \dfrac{1}{K+1} , & u = 1 . \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) , \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} , \quad u \neq 1 ,$$
$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little’s law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{\boldsymbol{R}} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

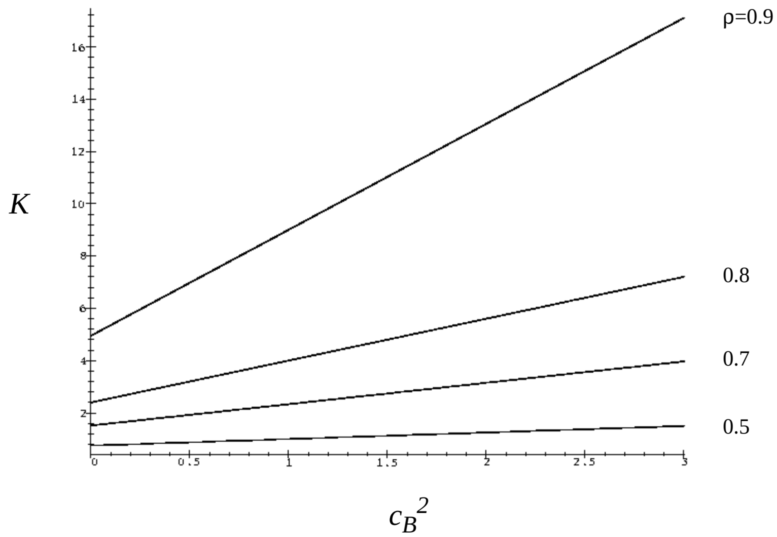
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

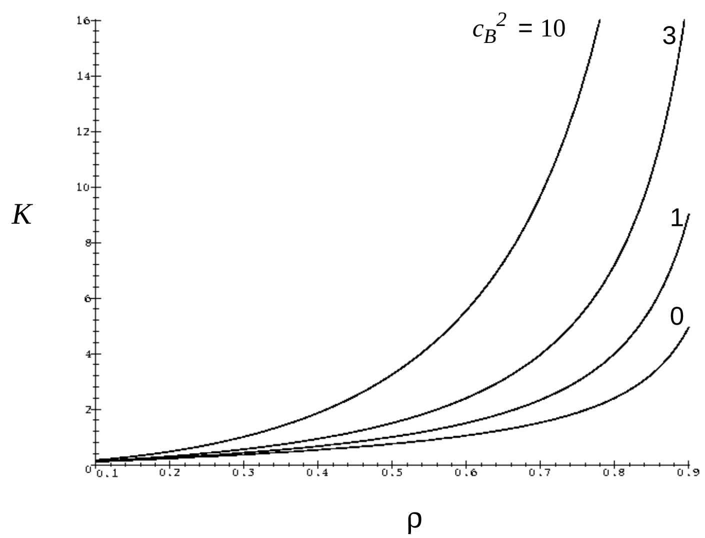
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

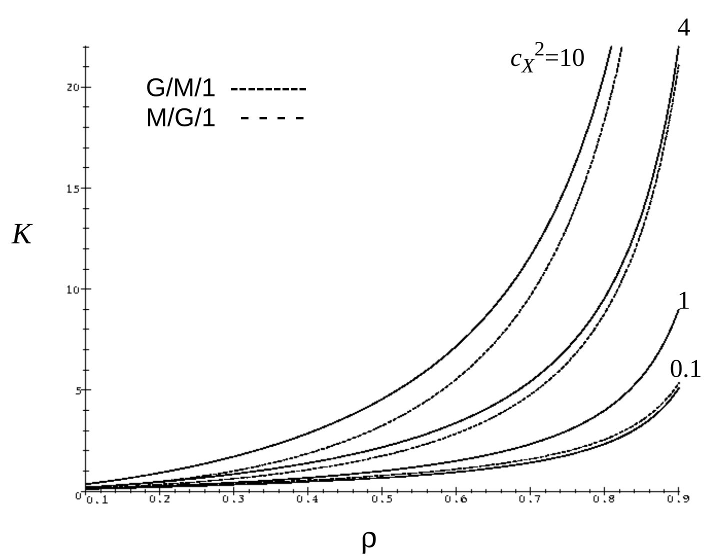
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

- Service time and interarrival time are not exponentially distributed
- No exact results
- The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

- Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

- Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

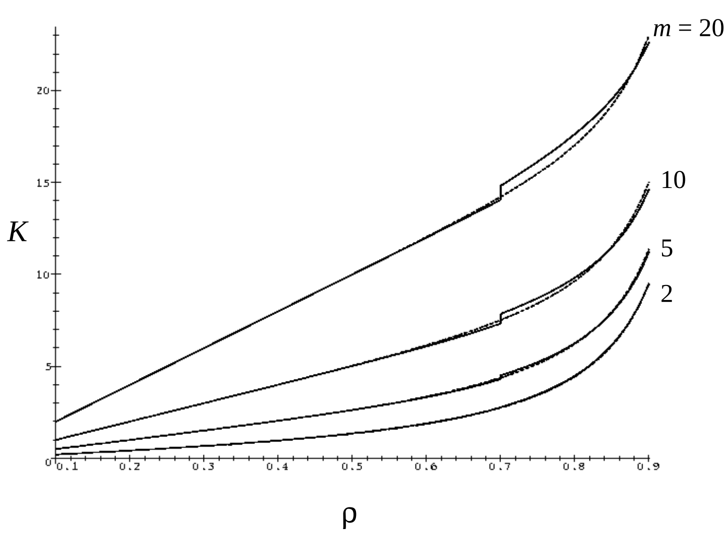
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7\,, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7\,. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

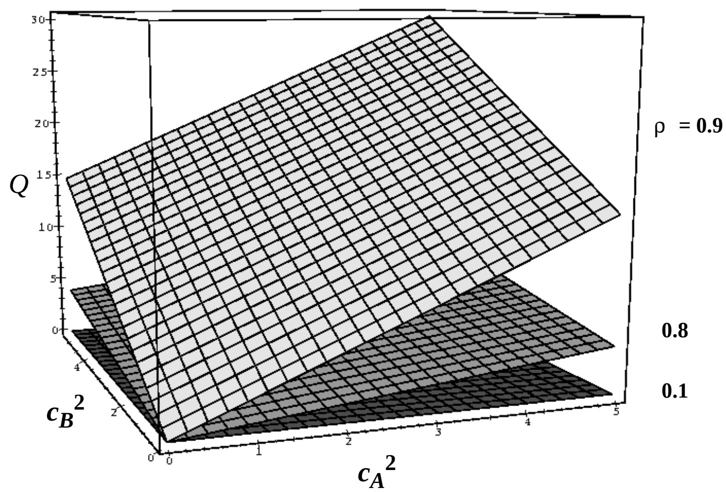
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

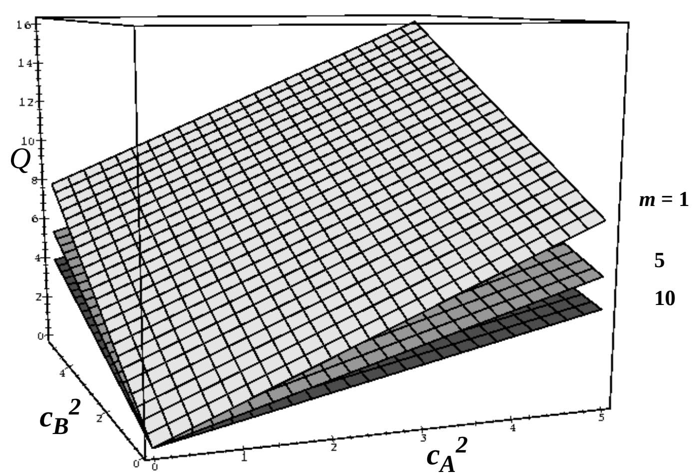
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\text{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\dfrac{2}{3}\dfrac{1-\rho}{P_m}\dfrac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\dfrac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



- Mean queue length $\overline{Q}$ of a G/G/10- system.

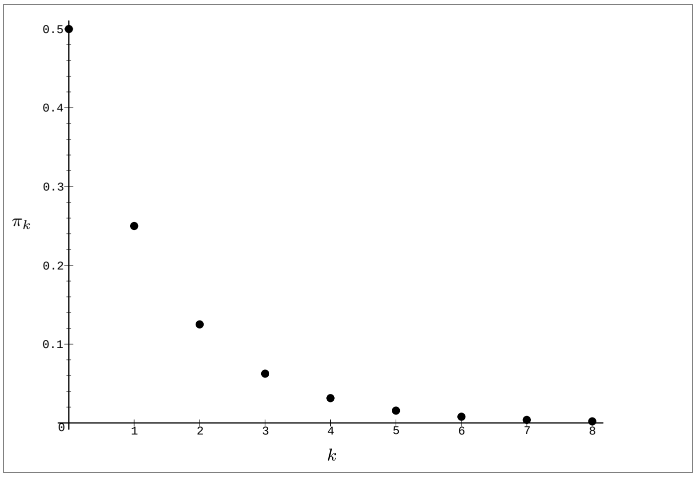


- Mean queue length $\overline{Q}$ of a G/G/m - system with $\rho = 0.8$

D.4 FIFO-Systems

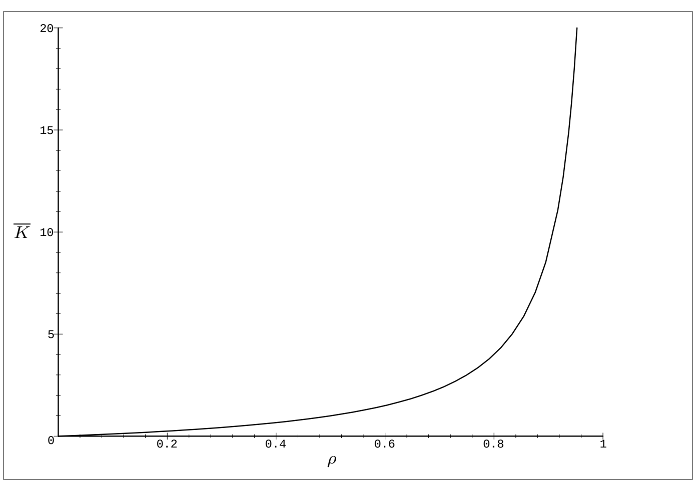
1 M/M/1

Steady-State Probability: $$\pi_k = (1-\rho)\rho^k$$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho} \right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k$$
$$= \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

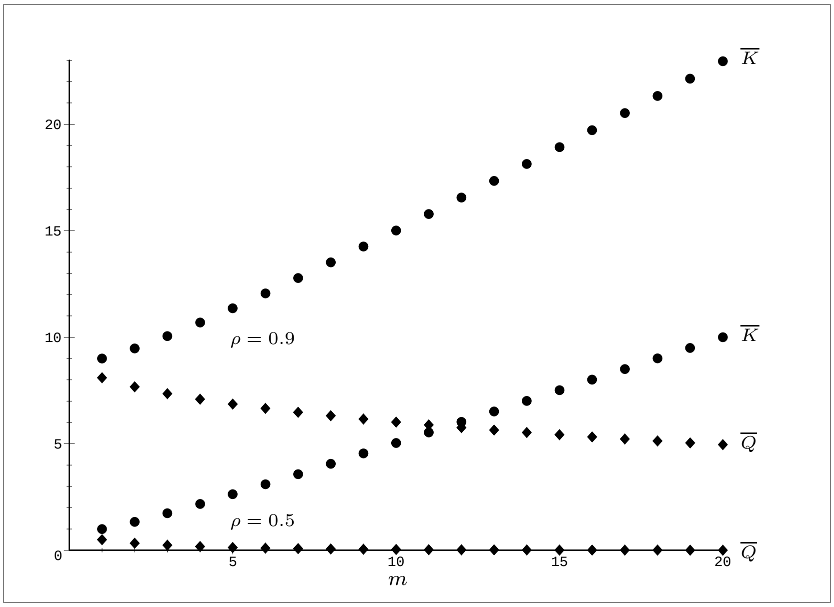
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \frac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \dots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

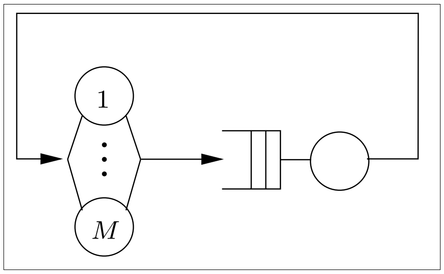
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

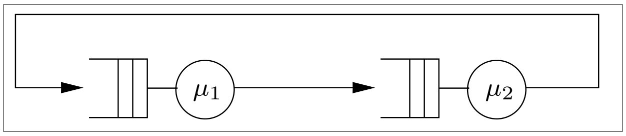
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1-\pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little's law):

$$\overline{K} = M - \frac{\mu(1-\pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* 1$)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\[2ex] \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2}\,,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}}\,, & u \neq 1\,, \\ \dfrac{1}{K+1}\,, & u = 1\,. \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0)\,, \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1)\cdot u^{K+1}}{1-u^{K+1}}\,, \quad u \neq 1\,,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

  We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

  And with Little's law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0$ = P(server is busy) $\overline{\boldsymbol{R}}$ + P(server is idle) 0

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System (c_B = 1) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

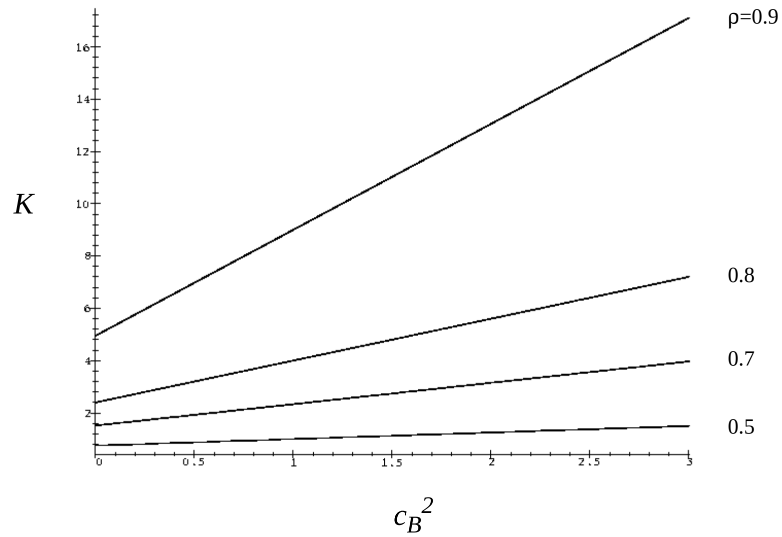
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems (c_B = 1) and M/D/1-systems (c_B = 0) we obtain:

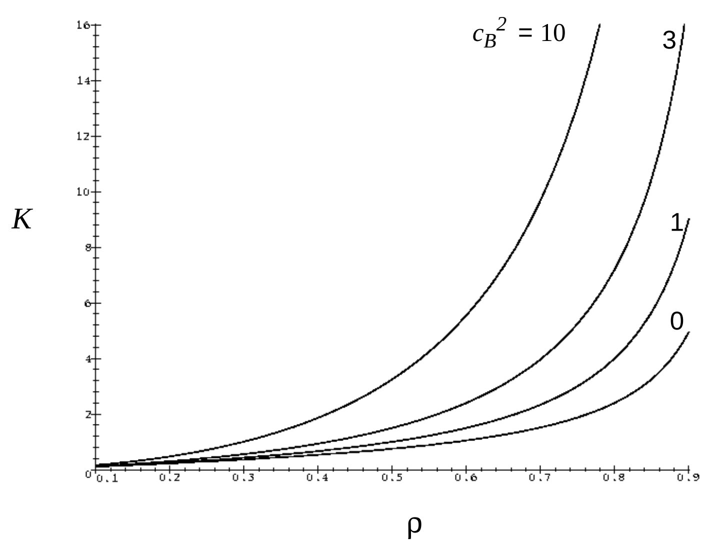
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

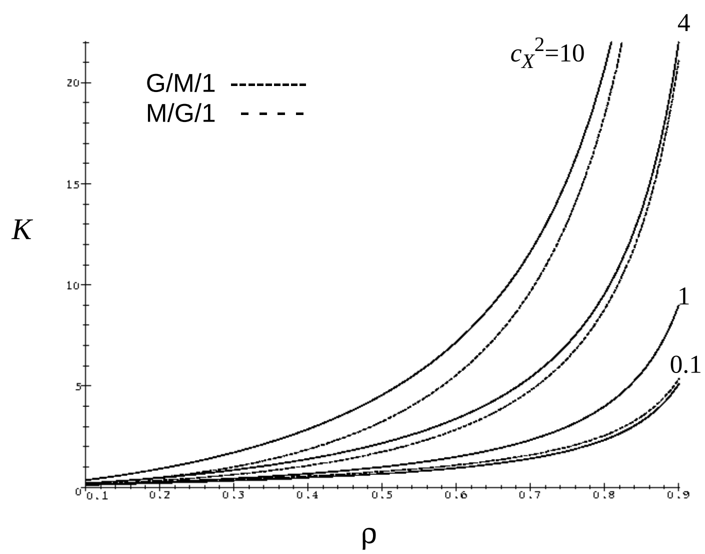
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
>1	>1	LB	LB
>1	<1	LB	UB
<1	>1	UB	LB
<1	<1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\mathrm{KLB}}$$

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

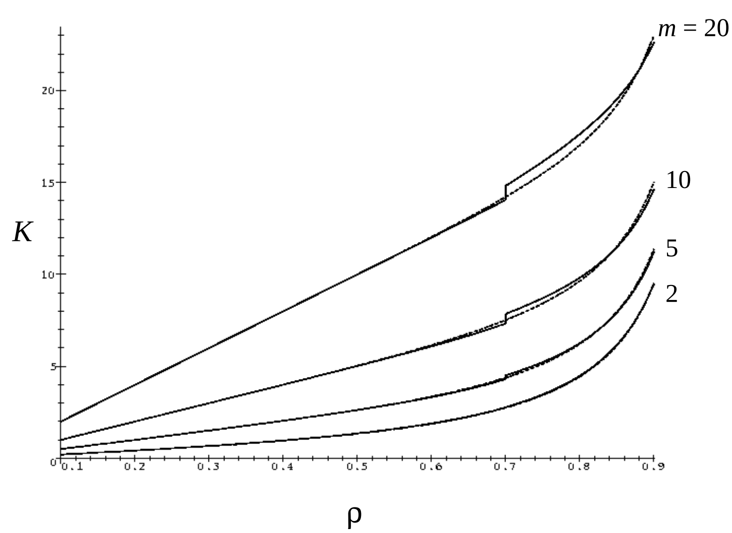
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

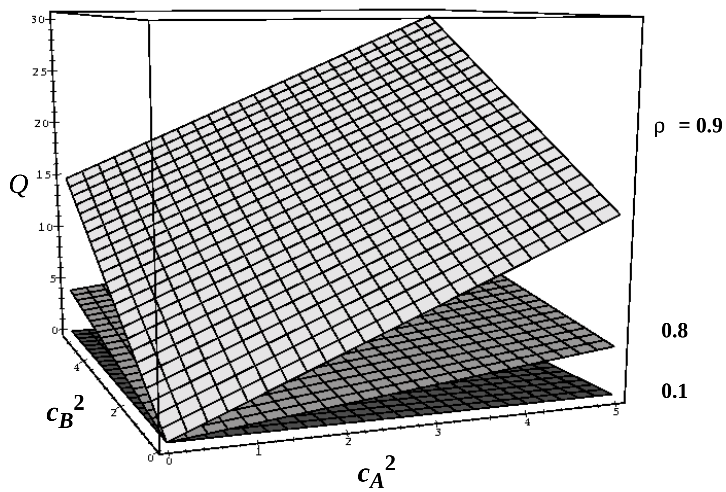
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

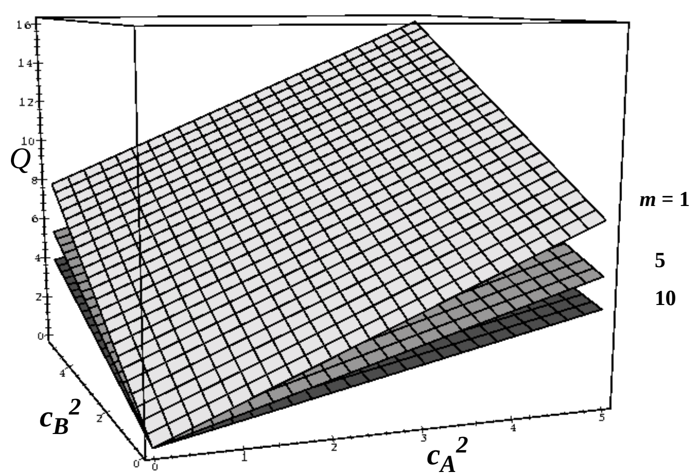
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



- Mean queue length $\overline{Q}$ of a G/G/10- system.

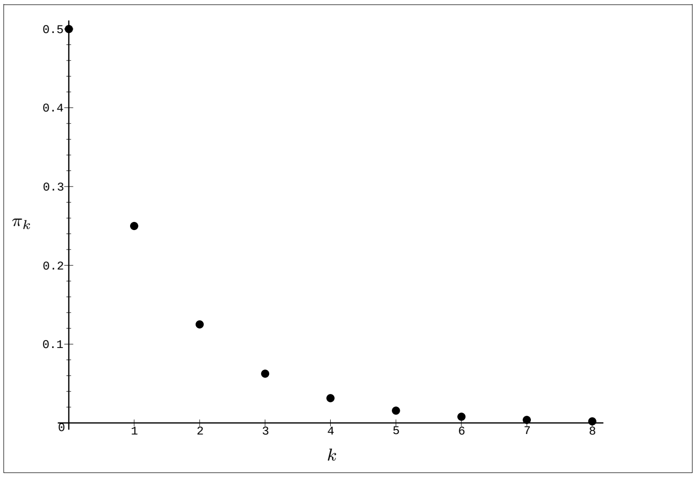


- Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8

D.4 FIFO-Systems

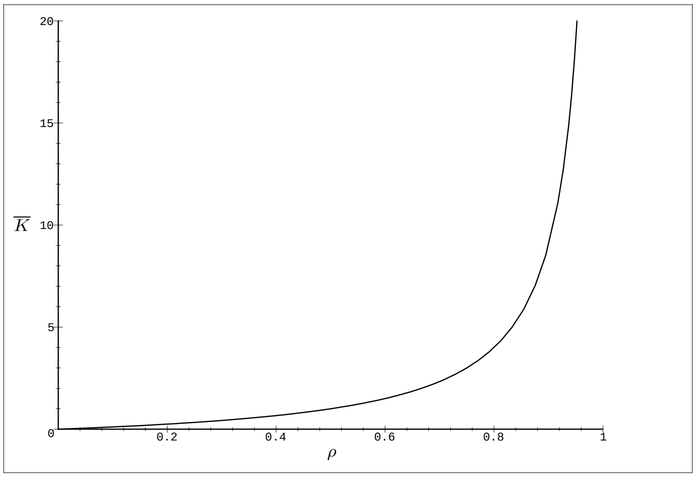
1 M/M/1

Steady-State Probability: $\pi_k = (1-\rho)\rho^k$



➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = \frac{\rho}{1-\rho}$$



➤ Variance of the mean number of jobs:

$$\sigma_K^2 = \frac{\rho}{(1-\rho)^2}$$

➤ Coefficient of variation of the mean number of jobs:

$$c_K = \frac{\sigma_K}{\overline{K}} = \frac{1}{\sqrt{\rho}}$$

➤ Distribution of the response time:

$$F_T(x) = 1 - \mathrm{e}^{-\mu(1-\rho)x}$$

➤ Variance of the response time:

$$\mathrm{var}(T) = \frac{1}{\mu^2(1-\rho)^2}$$

2 M/M/m

➤ Steady-state probability:

$$\pi_k = \begin{cases} \pi_0 \dfrac{(m\rho)^k}{k!}, & 0 \le k \le m \\ \pi_0 \dfrac{\rho^k m^m}{m!}, & k \ge m, \end{cases}$$

with:

$$\pi_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!} \frac{1}{1-\rho}\right]^{-1}$$

➤ Probability of waiting P_m :

$$P_m = P(K \geq m) = \sum_{k=m}^{\infty} \pi_k = \frac{(m\rho)^m}{m!(1-\rho)} \cdot \pi_0$$

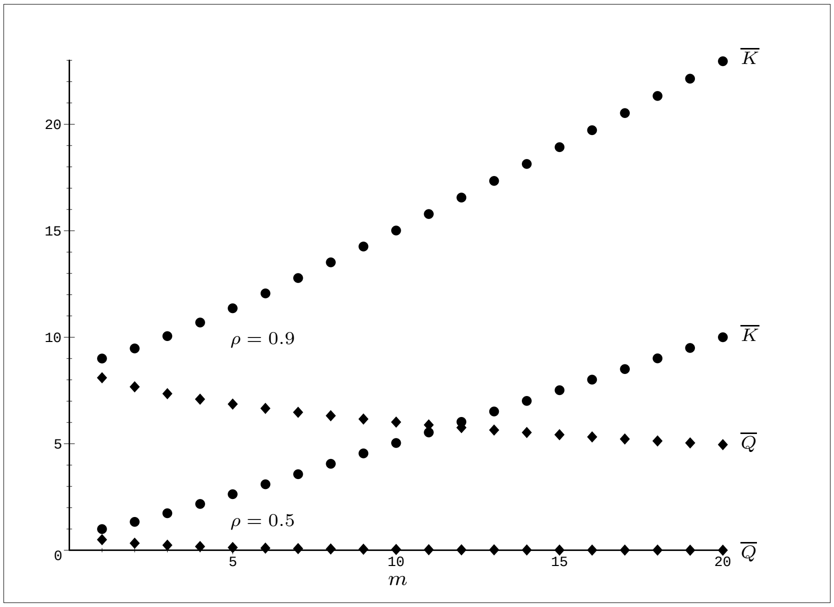
➤ Mean number of jobs $\overline{K}$:

$$\overline{K} = m\rho + \frac{\rho}{1-\rho} \cdot P_m$$

➤ Distribution of the waiting time:

$$F_W(x) = \begin{cases} 1 - P_m\,, & x = 0\,, \\ 1 - P_m \cdot \mathrm{e}^{-m\mu(1-\rho)x}\,, & x > 0\,. \end{cases}$$

➤ Mean queue length and mean number of jobs as functions of the number of servers:



3 M/M/oo-IS (Infinite Server)

- Infinite Server means, that the number of servers is at least the number of jobs in the system
- No job has to wait
- Steady-state probability:

$$\pi_k = \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} \cdot e^{-\frac{\lambda}{\mu}}$$

- Mean number of jobs:

$$\overline{K} = \frac{\lambda}{\mu}$$

- Mean response time = Mean service time:

$$\overline{T} = \frac{1}{\mu}$$

4 M/M/1/K Finite Capacity

- Queue has a finite capacity
- The maximum number of jobs in the system is K
- An arriving job is lost, if the queue is full
- Steady-state probability ($a = \lambda/\mu$):

$$\pi_k = \begin{cases} \dfrac{(1-a)a^k}{1-a^{K+1}}, & 0 \le k \le K, \\ 0, & k > K. \end{cases}$$

- For $\lambda = \mu$ ($a = 1$):

$$\pi_0 = \frac{1}{K+1} = \pi_k, \quad k = 1, 2, \ldots, K$$

➤ Mean number of jobs:

$$\overline{K} = \begin{cases} \dfrac{a}{1-a} - \dfrac{K+1}{1-a^{K+1}} \cdot a^{K+1}, & a \neq 1, \\ \dfrac{K}{2}, & a = 1. \end{cases}$$

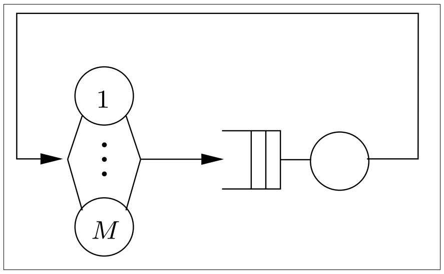
➤ Utilization:

$$\rho = 1 - \pi_0 \neq \lambda/\mu$$

➤ Throughput:

$$\lambda(1 - \pi_K) \neq \lambda$$

5 Machine-Repairman-Model



➤ Model for a system with M machines and a Repairman, who repairs a broken machine

➤ Model for a terminal system with M terminals and a computer

➤ Closed queueing network with M jobs

➤ Service rate ot the repairman: μ

➤ Failure rate of each machine: λ

➤ Steady-state probability (*P(k machines have to be repaired by the repairman)*):

$$\pi_k = \pi_0 \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}$$

with:

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k \frac{M!}{(M-k)!}}$$

➤ Utilization of the repairman:

$$\rho = (1 - \pi_0)$$

➤ Throughput:

$$\mu(1 - \pi_0)$$

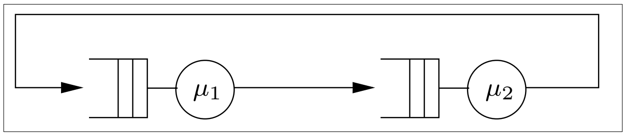
➤ Mean response time (Mean time which is needed until a machine is repaired):

$$\overline{T} = \frac{M}{\mu(1 - \pi_0)} - \frac{1}{\lambda}$$

➤ Mean number of jobs, mean number of broken machines (Little’s law):

$$\overline{K} = M - \frac{\mu(1 - \pi_0)}{\lambda}$$

6 Tandem-Network



- Closed network with K jobs
- State of the system: (k_1, k_2)

  with:

  - k_1: number of jobs in node 1
  - k_2: number of jobs in node 2
  - $k_1 + k_2 = K$

- Steady-state probability for node 1 ($P($ k_1 *jobs in node* $1)$):

$$\pi_1(k_1) = \pi_1(0) \cdot \frac{1}{u^{k_1}},$$

$$\pi_1(0) = \begin{cases} \dfrac{1 - \frac{1}{u}}{1 - \left(\frac{1}{u}\right)^{K+1}}, & u \neq 1, \\[2ex] \dfrac{1}{K+1}, & u = 1. \end{cases}$$

with:

$$u = \mu_1 / \mu_2$$

➤ Steady-state probability for node 2 (*P(k_2 Aufträge in Knoten 2)*):

$$\pi_2(k_2) = \pi_2(0) \cdot u^{k_2} ,$$

$$\pi_2(0) = \begin{cases} \dfrac{1-u}{1-u^{K+1}} , & u \neq 1 , \\ \dfrac{1}{K+1} , & u = 1 . \end{cases}$$

➤ Utilization of node 1 and node 2:

$$\rho_1 = 1 - \pi_1(0) , \qquad \rho_2 = 1 - \pi_2(0)$$

➤ Throughput:

$$\lambda = \lambda_1 = \lambda_2 = \rho_1\mu_1 = \rho_2\mu_2$$

➤ Mean number of jobs:

$$\overline{K}_2 = \frac{u}{1-u} - \frac{(K+1) \cdot u^{K+1}}{1-u^{K+1}} , \quad u \neq 1 ,$$

$$\overline{K}_1 = K - \overline{K}_2$$

7 M/G/1

- System is given by:
  - Interarrival time exponentially distributed with $\overline{T}_A = 1/\lambda$
  - Service time arbitrarily distributed:
    - Mean: $\overline{T}_B = 1/\mu$
    - Coefficient of variation: c_B
  - Number of servers: $m = 1$
  - Queueing discipline: FCFS

- The mean waiting time of an arriving job in a M/G/1 system has two components:
  - The mean remaining service time $\overline{W}_0$ of ther job in service (if any).
  - The sum of the mean service times of the jobs in the queue.

We can sum these components to:

$$\overline{W} = \overline{W}_0 + \overline{Q} \cdot \overline{T}_B$$

And with Little’s law: $\overline{Q} = \lambda\overline{W}$ und $\rho = \lambda/\mu$

$$\overline{W} = \frac{\overline{W}_0}{1-\rho}$$

◆ The mean remaining service time $\overline{W}_0$ is given by:

$\overline{W}_0 = P(\text{server is busy})\,\overline{\boldsymbol{R}} + P(\text{server is idle})\,0$

with the mean remaining service time $\overline{\boldsymbol{R}}$ of a busy server (the mean remaining service time of a idle server is obviously 0):

$$\overline{R} = \frac{\overline{T_B^2}}{2\overline{T}_B} = \frac{\overline{T}_B}{2}(1 + c_B^2)$$

For an M/M/1- System ($c_B = 1$) we obtain:

$$\overline{R}_{\mathrm{M/M/1}} = \overline{T}_B = \frac{1}{\mu}$$

◆ With Little's law ($\overline{Q} = \lambda\overline{W}$) we finally obtain:

$$\overline{Q} = \frac{\rho^2}{(1-\rho)} \cdot \frac{(1 + c_B^2)}{2}$$

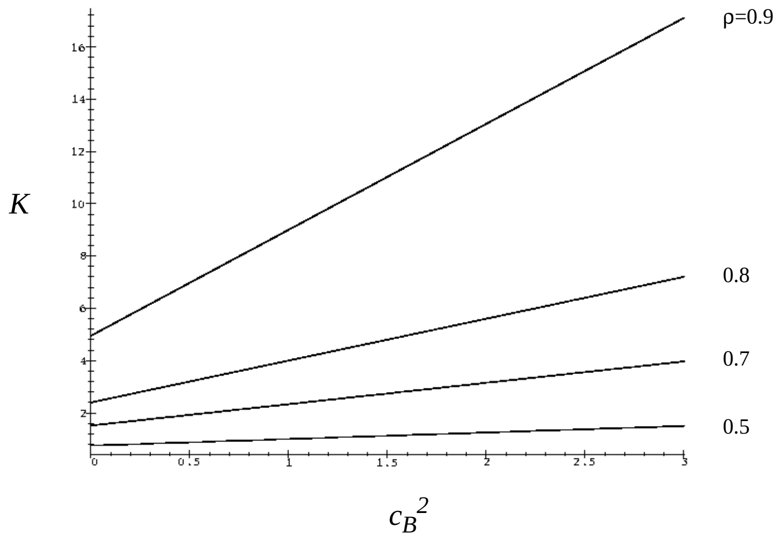
Pollaczek-Khintchine-Formula for the mean queue length of a M/G/1-system:

For M/M/1-systems ($c_B = 1$) and M/D/1-systems ($c_B = 0$) we obtain:

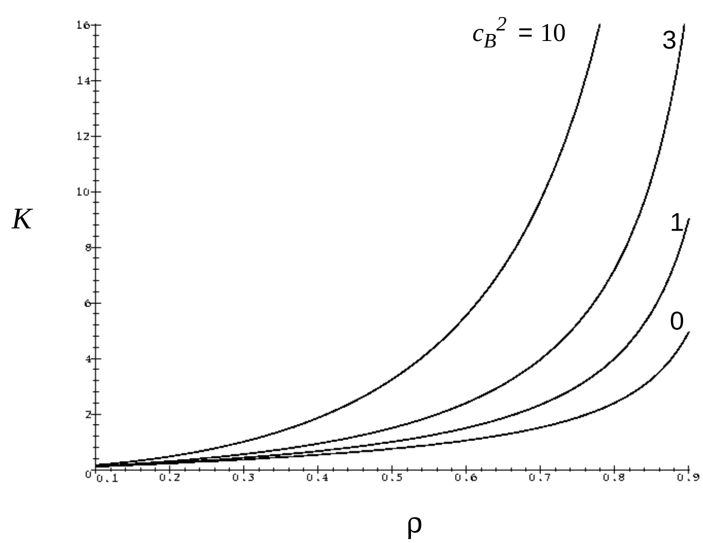
$$\overline{Q}_{\mathrm{M/M/1}} = \frac{\rho^2}{1-\rho}$$

$$\overline{Q}_{\mathrm{M/D/1}} = \frac{\rho^2}{2(1-\rho)}\,.$$

➤ Mean number of jobs as function of the coefficient of variation:



➤ Mean number of jobs as function of the utilization ρ:



8 G/M/1

◆ System parameters:

➤ Service time is exponentially distributed with $\overline{T}_B = 1/\mu$

➤ Number of servers: $m = 1$

➤ Interarrival time arbitrarily distributed with:

➤ Mean: $\overline{T}_A = 1/\lambda$

➤ Coefficient of variation: c_A

➤ Queueing discipline: FCFS

➤ Parameter σ:

$$\sigma = A^{\sim}(\mu - \mu\sigma)$$

$A^{\sim}$: Laplace transform of the pdf of the interarrival time.

➤ Mean number of jobs:

$$\overline{K} = \frac{\rho}{1-\sigma}$$

➤ Steady state probability:

$$\pi_k = \rho(1-\sigma)\sigma^{k-1}, \quad k > 0$$
$$\pi_0 = 1 - \rho\,,$$

◆ Example: M/M/1 - system:

$$A^{\sim}(s) = \lambda/(s+\lambda)$$

From this it follows:

$$\sigma = \frac{\lambda}{\mu} = \rho$$

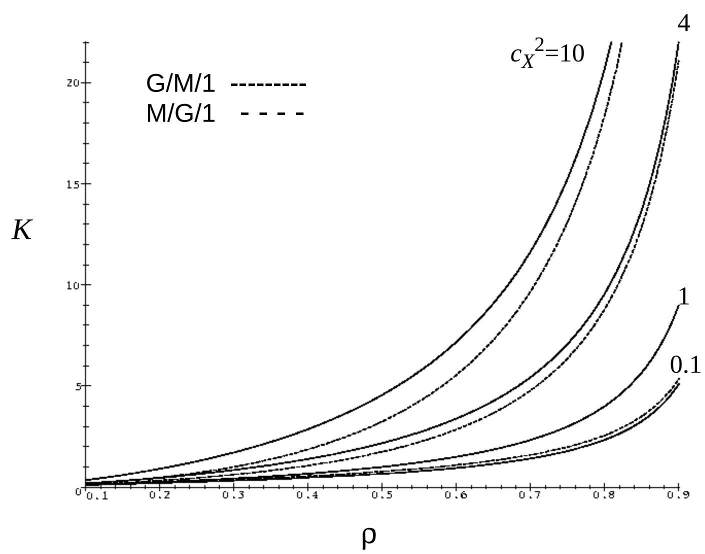
◆ Example: E_2/M/1 - system:

$$A^{\sim}(s) = \left(\frac{\lambda}{s+\lambda}\right)^2$$

It follows:

$$\sigma = \rho + \frac{1}{2} - \sqrt{\rho - \frac{1}{4}}$$

◆ Men number of jobs in a M/G/1- and a G/M/1- system:



9 G/G/1

➤ Service time and interarrival time are not exponentially distributed

➤ No exact results

➤ The formulas for the mean waiting time for M/G/1 bzw. G/M/1 - systems are lower and upper bounds for G/G/1 - systems, see table:

c_A^2	c_B^2	M/G/1	GI/M/1
> 1	> 1	LB	LB
> 1	< 1	LB	UB
< 1	> 1	UB	LB
< 1	< 1	UB	UB

LB: Lower Bound

UB: Upper Bound

➤ Allen-Cunneen approximation formula for the mean waiting time:

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2}$$

Exact for M/M/1 und M/G/1

➤ Krämer/Langenbach-Belz approximation formula (very accurate):

$$\overline{W} \approx \frac{\rho/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2} \cdot G_{\text{KLB}}$$

$$G_{\text{KLB}} = \begin{cases} \exp\left(-\frac{2}{3} \cdot \frac{1-\rho}{\rho} \cdot \frac{(1-c_A^2)^2}{c_A^2 + c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2 - 1}{c_A^2 + 4c_B^2}\right), & c_A > 1 \end{cases}$$

10 M/G/m

➤ Allen-Cunneen approximation formula:

$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{(1+c_B^2)}{2m}$$

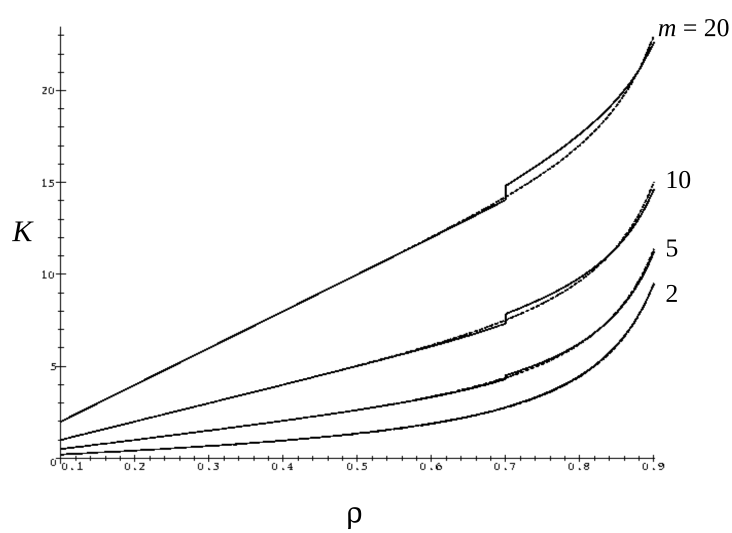
Approximation for the probability of waiting: $P_{m,M/M/m}$, or:

$$P_m \approx \begin{cases} \dfrac{\rho^m + \rho}{2}, & \rho > 0.7, \\ \rho^{\frac{m+1}{2}}, & \rho < 0.7. \end{cases}$$

m = 5:

ρ	0.2	0.4	0.6	0.7	0.8	0.9	0.95	0.99
$P_{m_{ex}}$	0	0.06	0.23	0.38	0.55	0.76	0.88	0.97
$P_{m_{app}}$	0	0.06	0.21	0.34	0.56	0.75	0.86	0.97

Mean number of jobs in a M/M/m - system with $P_{m,ex}$ und $P_{m.appr}$.



11 G/G/m

➤ Allen-Cunneen approximation formula for the mean waiting time:

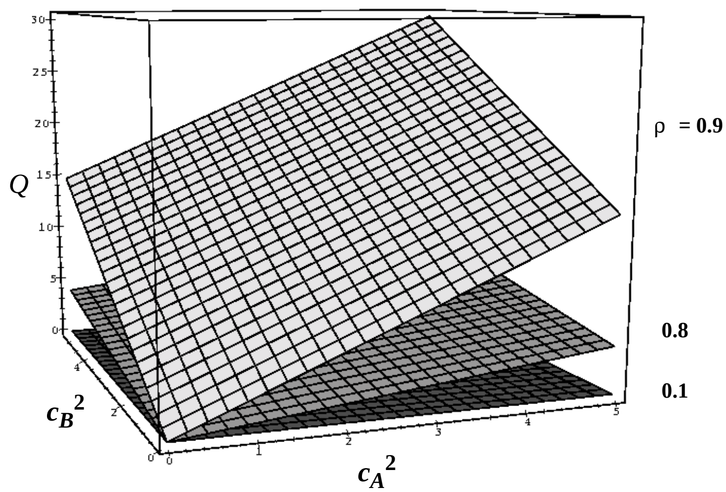
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m}$$

➤ Krämer/Langenbach-Belz approximation formula for the mean waiting time:

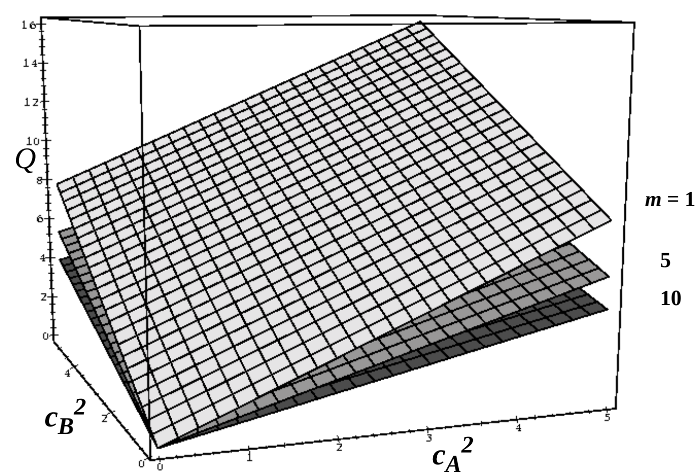
$$\overline{W} \approx \frac{P_m/\mu}{1-\rho} \cdot \frac{c_A^2 + c_B^2}{2m} \cdot G_{\mathrm{KLB}}$$

➤ Correktion factor G_{KLB}:

$$G_{\mathrm{KLB}} = \begin{cases} \exp\left(-\frac{2}{3}\frac{1-\rho}{P_m}\frac{(1-c_A^2)^2}{c_A^2+c_B^2}\right), & 0 \le c_A \le 1 \\ \exp\left(-(1-\rho)\frac{c_A^2-1}{c_A^2+4c_B^2}\right), & c_A > 1, \end{cases}$$



• Mean queue length $\overline{Q}$ of a G/G/10- system.



• Mean queue length $\overline{Q}$ of a G/G/m - system with ρ = 0.8